# The Concept of Contiguity in Models Based on Time-Indexed Formulations

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**Abstract.** This paper describes an extension which can be added to any production planning model based on time-indexed formulations with at most one setup- or mode-change per period. It allows to add constraints involving accumulated quantities over several time-slices implementing the concept of contiguity into the model. This feature is relevant to any kind of process industry. It allows to model batch and campaign production, or to require that the minimum time between mode-changes must be larger than a specified lower bound. The key idea used in the technical approach is to identify which time-indexed quantities belong to certain contiguous components, *e.g.*, campagains, over several time slices and to replace products of continuous variables and binaries, or absolute value terms by linear relations involving additional binary variables. In this contribution we apply this approach to a special case and extend a production planning problem in chemical industry.

#### 1 Introduction

The motivation for the model extension has its root in batch or campaign production in the chemical process industry. Batch production in the process industry operates in integer multiples of batches where a batch is the smallest unit to be produced, *e.g.*, 200 tons. Several batches following each other immediately establish a campaign. The production may be subject to certain restrictions, *e.g.*, campaigns are built up by a discrete number of batches, or that only campaigns of a minimal size can be produced. Within a fixed planning horizon, T, a certain product can be produced in several campaigns.

To be as general as possible let us consider batch reactors which can be, for example, operated in different modes producing several products in each mode with different free or fixed recipes leading to a general mode-product relation ([5], pp.153-155, 320-324). Thus, in a certain mode several products are produced (with different daily production capacity rates), and vice-versa, a product can be produced in different modes. Daily production can be less then the capacity rates. This is an important feature in coupled production if the production planning problem is demand driven.

In the context of time-indexed formulation where variables  $p_{pt}$  describe the production [e.g., in tons] of a product p in period (time-interval) t it is not easy to model batch or campaign restrictions if the batch or minimal campaign size is larger than the capacity per period. Assume that production is performed in batches of 200 tons, and that our time intervals have a length of ten days with a daily production rate of 10 tons/day. The minimum time to produce the batch would cover 20 days, or exactly two time intervals. A plan looking like  $p_{p4} = 45$  tons,  $p_{p5} = 100$  tons, and  $p_{p6} = 55$  tons covers three periods (the first and third only partial) to produce exactly 200 tons, and thus provides more degrees of freedom.

Brockmüller and Wolsey [3] solved the problem for a special case (production equals the capacity rates). Their approach uses explicitly the feature that production equals the capacity rates in order to compute a priori the number of periods to produce a campaign of specified minimal size.

However, if daily production can take any value between a lower bound, *e.g.*, zero and the capacity rate, or if a product is produced, for example, according to general mode-product relation, then this a priori information is not available.

Our approach does not depend on this a priori information and can be used for more general cases. It is also possible to model the requirement that a certain time-lag between successive mode-changes is observed.

# 2 Modeling Contiguity

To model contiguous quantities, *e.g.*, the amount of production in a certain campaign extending over several time slices, in the framework of time-indexed formulations we proceed as follows:

- 1. we identify or assign a time-indexed quantity, *e.g.*, a production variable, contributing to a contiguous quantity, *e.g.*, the production in a campaign, uniquely to a contiguous quantity (this requires that we trace the start-ups of production in the time-slices, and the start-ups of the contiguous quantities)
- 2. we add the values of all contributing quantities belonging to a certain contiguous quantity (we use indication variables which tell us whether a certain product amount contributes to a certain campaign)
- 3. we apply constraints to the contiguous quantities

#### 2.1 Continuous Production Variables and State Variables

An important intermediate goal is to compute the amount,  $p_{rpnt}^C$ , of product  $p \in \mathcal{P}$  produced for a certain campaign n in period  $t \in \mathcal{T}$ . Thus, our idea is to relate the production  $p_{rpt}$  to a certain campaign and then to add all contributions establishing this campaign.

The mathematical model, for this class of production planning problems in the process industries, is assumed to be, for a certain site, unit, or reactor  $r \in \mathcal{R}$ , based on some binary state variables  $\delta_{rpt}^{P}$  indicating whether product p is produced on r in period t, and binary start-up variables  $\delta_{rpt}^{S}$  indicating

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whether the production of p is started in period t on r. Let  $P_{rpt}^-$  and  $P_{rpt}^+$  be bounds on  $p_{rpt}$  if  $p_{rpt} > 0$ . We may choose the upper bound  $P_{rpt}^+$  for  $p_{rpt}$ , e.g., as the length of the period (in days) times the daily production capacity, and the lower conditional bound  $P_{rpt}^- = 0.8P_{rpt}^+$ . Let us, at first, connect  $\delta_{rpt}^P$  to the production variables  $p_{rpt}$  starting with

the inequalities

$$p_{rpt} \le P_{rpt}^+ \delta_{rpt}^P$$
 ,  $\forall \{rpt\}$  . (1)

If  $\Delta_{rp}$  tells us whether product p is produced at the beginning of the first period, and  $\sum_{p} \Delta_{rp} = 1$ , then for the first period we have

$$P_{rp1}^{-}\delta_{rp1}^{P} - P_{rp1}^{+}(1 - \Delta_{rp}) - P_{rp1}^{+}\left(1 - \delta_{rp2}^{P}\right) \le p_{rp1}, \quad \forall \{rp\} \quad , \qquad (2)$$

and for all other period (except the last one)  $\mathcal{T}_1 := \{2, \ldots, T-1\}$ 

$$P_{rpt}^{-}\delta_{rpt}^{P} - P_{rpt}^{+}\delta_{rpt}^{S} - P_{rpt}^{+}\left(1 - \delta_{rpt+1}^{P}\right) \le p_{rpt}, \quad \forall \{rpt \in \mathcal{T}_{1}\} \quad . \tag{3}$$

The inequalities (1) to (3) hold the positivity conditions ( $\delta_{rpt}^P = 0 \Leftrightarrow p_{rpt} = 0$ ) and  $(\delta_{rpt}^{P} = 1 \Leftrightarrow P_{rpt}^{-} \leq p_{rpt} \leq P_{rpt}^{+})$  for all inner periods of a campaign. The second and third term on the left-hand side of (3) ensure that the positivity conditions is not applied to the first and last period of campaigns.

#### State Variables and Start-Up Variables 2.2

Now we need to relate the start-up variables to the state variables. This part depends on the problem considered. A formulation, valid for any continuous variable (e.g., the production variable  $p_{rpt}$  or the variable  $m_{rmt}^D$  denoting the time spent in mode m both used in [5], Section 10.4; we refer to that model as M1 from now on) subject to constraints cross periods, and the conditions that we can produce only one product per time and that at most two products can be produced during one period (*i.e.*, at most one setup-change per period), needs to represent the following set of implications for  $\delta_{rpt}^S$ :

$$\begin{array}{c|cccccc} \delta^{P}_{rpt-1} \setminus \delta^{P}_{rpt} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & \mu_{rpt} \end{array} , \qquad (4)$$

with

$$\mu_{rpt} := \begin{cases} 1, \text{ if any other production } p' \neq p \text{ is started in period } t-1 \\ 0, \text{ if no other production started in period } t-1 \end{cases}$$
(5)

These rules, for  $\delta^P_{rpk} + \delta^P_{rpk-1} \neq 2$  are enforced by

$$\delta_{rpt}^S = \delta_{rpt}^P \quad , \quad \forall \{rp\} \quad , \quad t = 1 \quad , \tag{6}$$

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for the first period, and for all other periods  $\mathcal{T}_T := \{2, \ldots, T\}$  by

$$\delta_{rpt}^{S} \leq \delta_{rpt}^{P} \quad , \quad \delta_{rpt}^{S} \geq \delta_{rpt}^{P} - \delta_{rpt-1}^{P} \quad , \quad \forall \{rpt \in \mathcal{T}_{T}\} \quad . \tag{7}$$

The case  $\delta^P_{rpk-1} = \delta^P_{rpt} = 1$  is properly described by additional inequalities

$$\delta_{rpt}^{S} \ge -2 + \sum_{p' \neq p} \delta_{rp't-1}^{S} + \delta_{rpt}^{P} + \delta_{rpt-1}^{P} \quad , \quad \forall \{rpt \in \mathcal{T}_{T}\} \quad , \tag{8}$$

and

$$\delta_{rpt}^{S} \le 2 + \sum_{p' \ne p} \delta_{rp't-1}^{S} - \delta_{rpt}^{P} - \delta_{rpt-1}^{P} \quad , \quad \forall \{rpt \in \mathcal{T}_{T}\} \quad . \tag{9}$$

If, in a general mode-product relation several products can be produced simultaneously, we may want that the case  $\delta_{rpt-1}^P = \delta_{rpt}^P = 1$  also leads to  $\delta_{rpt}^S = 0$ ; this can easily be realized by neglecting the second term on the right-hand sides of (8) and (9). Alternatively, we may require that  $\mu = 1$  if any other more complicated rule than the above (6) is fulfilled.

## 2.3 Counting and Identification of Campaigns

Let us from now on assume that  $\delta^P_{rpt}$  and  $\delta^S_{rpt}$  are available. The production of product p may start in several time periods, *i.e.*, we have several product-p-campaigns within the planning horizon T. Therefore we introduce continuous variables,  $c_{rpt} \geq 0$ , counting the number of start-ups and related to the start-up variables  $\delta^S_{rpt}$  by

$$c_{rp1} = \delta_{rp1}^S$$
,  $\forall \{rp\}$ ;  $c_{rpt} = c_{rpt-1} + \delta_{rpt}^S$ ,  $\forall \{rpt \in \mathcal{T}_T\}$ . (10)

Now we introduce continuous variables  $\nu_{rptn}$  indicating whether a certain campaign,  $n \in \mathbb{N}_0$ , could be active  $(c_{rpt} = n)$  or not, *i.e.*, whether  $c_{rpt}$  is equal to a certain fixed integer  $n \in \mathbb{N}_0$ , or not.  $\nu_{rptn}$  represents the nonlinear function

$$\nu_{rptn} = 1 - \theta \left( |c_{rpt} - n| \right) \quad , \quad \theta(x) := \begin{cases} 1, \text{ if } x > 0\\ 0, \text{ if } x = 0 \end{cases} \quad . \tag{11}$$

Let us assume that at most  $N_{rp}^+ \in \mathbb{N}_0$  campaigns of product p can be produced within the planning horizon; a typical value in the current planning problem is  $N_{rp}^+ = 6$ . A special case is  $N_{rp}^+ = 1$  enforcing that a product can be produced in only one campaign.

The relation (11) is enforced by

$$1 = \sum_{n=0}^{N_{rp}^{+}} \nu_{rptn} \quad \text{and} \quad \sum_{n=0}^{N_{rp}^{+}} n\nu_{rptn} = c_{rpt} \quad , \quad \forall \{rpt\} \quad , \tag{12}$$

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*i.e.*, one campaign has to be chosen in any case (possibly, the "0" campaign), and if campaign n is selected then  $c_{rpt} = n$ . The sets

$$S_{rpt} := \left\{ \nu_{rptn} \left| 0 \le n \le N_{rp}^+ \right\} \quad , \quad \forall \{rpt\}$$

$$\tag{13}$$

form a special ordered set of type 1. We use the second equation of (12) as the reference row for efficient branching.

The total amount,  $p_{rpn}^C$ , of product p produced within campaign n is given by

$$p_{rpn}^{C} = \sum_{t=1}^{N^{*}} p_{rptn}^{C} , \quad \forall \{rp\} , \quad \forall n \in \mathcal{N}_{1} := 1, \dots, N_{rp}^{+} , \qquad (14)$$

where  $p_{rptn}^C$  is the amount of product p produced for campaign n in period t, *i.e.*,

$$p_{rptn}^C = p_{rpt}\nu_{rptn} \quad , \quad \forall \{rptn \in \mathcal{N}_1\} \quad . \tag{15}$$

Applying the formalism described in Section 3 with K = 1 we replace (15) by

$$p_{rptn}^{C} \leq P_{rpt}^{+} \nu_{rptn} \quad , \quad p_{rptn}^{C} \leq p_{rpt} \quad ,$$
  
$$p_{rptn}^{C} \geq p_{rpt} - P_{rpt}^{+} + P_{rpt}^{+} \nu_{rptn} \quad , \quad \forall \{rptn\} \quad .$$
(16)

With the formalism at hand described above we reached our goal: the computation of the amount,  $p_{rpn}^C$ , of product p produced for campaign n.  $p_{rpn}^C$ may be now subject to specific batch or campaign constraints.

#### 2.4 Batch and Campaigns Conditions

We are now in a position to formulate that, *e.g.*, a campaign may just consist of one single batch of fixed batch size  $B_{rp}$ ,

$$p_{rpn}^C = B_{rp} \quad , \quad \forall \{rpn \in \mathcal{N}_1\} \quad . \tag{17}$$

Alternatively, campaigns may be built up by a discrete number of batches following each other immediately, *i.e.*,

$$p_{rpn}^C = B_{rp}\beta_{rpn} \quad , \quad \forall \{rpn \in \mathcal{N}_1\} \quad , \tag{18}$$

where the integer variable  $\beta_{rpn}$  indicates the number of batches of size  $B_{rp}$  within campaign n. Finally,  $p_{rpn}^C$ , may behave like a semi-continuous variables, *i.e.*,

$$p_{rpn}^C = 0 \quad \text{or} \quad C_{rp}^- \le p_{rpn}^C \le C_{rp}^+ \quad , \quad \forall \{rpn \in \mathcal{N}_1\} \quad ,$$
 (19)

where  $C_{rp}^{-}$  and  $C_{rp}^{+}$  are lower and upper bounds if production takes place.

## 3 Modeling Product Terms Including One Continuous & Several Binary Variables

To model products like  $x\Pi_{k=1}^{K}\delta_k$ , where  $\delta_k$  are binary variables and x is any kind of non-negative variable, let us assume that  $X^+$  is a valid upper bound on x. The product  $\Pi_{k=1}^{K}\delta_k$  is exactly represented by the variable y subject to the inequalities

$$\forall k : y \le X^+ \delta_k \quad , \quad y \le x \quad , \quad y \ge x - X^+ \left( K - \sum_{k=1}^K \delta_k \right) \quad . \tag{20}$$

The first inequality of (20) has the implications ( $\delta_k = 0 \Rightarrow y = 0$ ) and  $(y > 0 \Rightarrow \sum_{k=1}^{K} \delta_k = K)$ , while the second and third inequality give us  $(\sum_{k=1}^{K} \delta_k = K \Rightarrow y = x)$  and  $(y = 0 \Rightarrow \sum_{k=1}^{K} \delta_k < K)$ . Note that if we want to know the product  $y = x \prod_{k=1}^{K} \delta_k$  explicitly we do not need to introduce an extra variable.

#### 4 Implementation and Results

If we want to add the batch constraints in Section 2 to the production planning model M1<sup>1</sup>, it is not strictly necessary to use (2)-(9) to compute  $\delta_{rpt}^P$ and  $\delta_{rpt}^S$ . Alternatively, we can derive  $\delta_{rpt}^P$  and  $\delta_{rpt}^S$  from the mode state variables  $\alpha_{rmt}$  and start-up variables  $\beta_{rmt}$  used in the model M1 by [4] and ([5], pp.320-324). If  $\mathcal{P}$  is the union of disjunctive sets  $\mathcal{P}_m$  of products produced in mode m and  $I_{rmp}$  indicates whether product p can be produced in mode m on reactor r, (in the current case we have even  $\sum_p I_{rmp} = 1$ , *i.e.*, exactly one product per mode) we just have

$$\delta_{rpt}^P = \sum_{m|I_{rmp}=1} \alpha_{rmt} \quad , \quad \delta_{rpt}^S = \sum_{m|I_{rmp}=1} \beta_{rmt} \quad , \quad \forall \{rpt\} \quad . \tag{21}$$

This special approach based on (21) is, however, only exactly identical with the more general approach based on (6)-(9) if  $P_{rpt}^- = 0$ .

For the model M1 and a typical reference scenario  $(S_1)$  covering 12 to 36 production time periods we have used both approaches indicated by indices s and g to derive production plans maximizing total sales. The scenarios  $S_2$ use (19) to model campaigns whose minimum size is 300 tons. The scenarios  $S_3$  include 49 partial integer variables and use (18) to enforce that campaigns are built up by discrete batches of 100 tons each. Finally, in scenario  $S_m$  we require that if a certain mode is chosen the plant has to stay in that mode

<sup>&</sup>lt;sup>1</sup> Although the problem instance specified by the data leads to a one-to-one relation between modes and products, the coupling  $p_{rpt} \leq \sum_m R_{mp} m_{rmt}^D$  with production rates  $R_{mp}$  holds for any mode-product relation. Note that in M1  $r \leftrightarrow i$ , and  $t \leftrightarrow k$ .

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for at least 3 days. In this case, the variables  $m_{imk}^D$  used in ([5], pp.320-324), expressing how much time the plant at site *i* spends in mode *m* in period *k*, play the role of  $p_{rpt}$  used above; the length of the period (10 to 30 days) is a useful upper bound on  $m_{imk}^D$ . Using Dash's MILP-solver XPRESS-MP 10.05 ([1], [2]) we got the results in Table 1 (including the number of continuous, binary and semi-continuous variables, constraints, integer solution, number of nodes  $n_n$ , running time  $\tau$ , and gap  $\Delta$  in percent) when we applied the formalism to all possible reactor(site)-product(mode)-time combinations.

	$P_{rpt}^{-}$	$n_c$	b	s-c	c	IP	$n_n$	au	Δ
$S_1$		12397	2973	1608	8441	1	440	$8^{\mathrm{m}}$	1.9
$S_1$	_					2	960	$+6^{\mathrm{m}}$	1.4
$S_1$	—					3	1721	$+8^{\mathrm{m}}$	1.0
$S_{2s}$	—	14833	2973	1650	13997	1	786	$52^{\mathrm{m}}$	24.1
$S_{2g}$	1	15217	2973	1650	14033	1	858	$59^{\rm m}$	40.5
$S_{2g}$	$0.8P_{rpt}^+$	15217	2973	1650	14033	2	3860	$5^{\rm h}59^{\rm m}$	46.9
$S_{3s}$		15687	2973	1608	13855	1	632	$58^{\rm m}$	4.1
$S_{3g}$	1	16215	2973	1608	15681	1	907	$1^{\rm h}09^{\rm m}$	19.7
$S_{3g}$	$0.8P_{rpt}^+$	16215	2973	1608	15681	3	18272	$39^{\rm h}02^{\rm m}$	4.8
$S_m$		15333	2973	1650	15681	1	511	$28^{\rm m}$	4.9
$S_m$						3	1943	$+40^{m}$	1.8
$S_m$	—					4	3972	$+2^{\rm h}44^{\rm m}$	1.6

Table 1. Experimental Test Runs

The gap  $\Delta$  listed in Table 1 has been calculated as

$$\Delta = 100 \frac{z^{UB} - z^{IP}}{z^{IP}} = 100 \left(\frac{z^{UB}}{z^{IP}} - 1\right)$$
(22)

The upper bound,  $z^{UB}$ , unfortunately stays constant in the whole B&B tree and has always the value  $z^{UB} = 54479$ . The use of special ordered sets of type 1 for the variables  $\nu_{rptn}$  is essential; the model contains 192 sets and 1080 set members. In previous versions when these variables were declared as binary variables computing times were much larger. In the  $S_3$  runs (multiple batches), the 49 variables  $\beta_{rpn}$  were declared as partial integers (integer below 10, continuous above 10). Although the variables  $\delta_{prt}^S$  become binary automatically, it is advantageous to declare them as binary variable explicitly because that enables us to prioritize them and to improve branching. The use of directives in the model was crucial. In the general approach scenarios  $n_n$ ,  $\tau$ , and the quality of the solution indicated by  $\Delta$  depended critically on  $P_{rpt}^-$ . Note that the run for  $P_{rpt}^- = 0.8P_{rpt}^+$ , (*i.e.*, high utilization rates of the plant system), shows the third integer solution found and required much more computing time.

## 5 Summary and Conclusions

The benefit achieved by the extended model features for the special production planning problem discussed is of qualitative nature because it leads to an improved representation of the real world process. The production plans do not suffer any longer from the time-indexed formulation and look more stable avoiding small campaigns and many setup-changes. In practical planning runs it is sufficient to use the formalism only for a few products or modes, and sometimes only for one site or reactor. Thus, the Pentium 166 MHz computing time reduces to less than 15 minutes and becomes similar to the one of the reference scenario  $S_1$ .

Future direction regarding the problem will focus on special branching rules and cuts to improve the gap, and to reduce the upper bound. For the current application this is not a serious problem because only a few products required constraints across period which leads to reasonable running time and a gap of at most a few percent.

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