THE ASTROPHYSICAL JOURNAL, 313:346–357, 1987 February 1

A NEW METHOD TO OPTIMIZE PARAMETERS IN SOLUTIONS OF ECLIPSING BINARY LIGHT CURVES

JOSEF KALLRATH
Sternwarte der Universität Bonn

AND

ALBERT P. LINNELL
Department of Physics and Astronomy, Michigan State University
Received 1986 June 19; accepted 1986 July 22

ABSTRACT

Optimization procedures currently in use in the solution of eclipsing binary light curves require calculation of partial derivatives of received light with respect to the system parameters. Large amounts of computer time may be required for these calculations. Complications often arise in achieving convergence with differential correction techniques. These complications may result from an initial parameter set that is too remote from the correct solution, inaccurate calculated derivatives, parameter correlations, or an ill-conditioned set of normal equations.

We present an optimization procedure, the simplex algorithm, which cannot diverge, requires no calculated partial derivatives, is capable of automatic iteration, is operationally stable, and is computationally efficient. An application to MR Cygni data, using individual observations rather than normal points, illustrates the operation of the method.

Subject headings: numerical methods — stars: eclipsing binaries — stars: individual

I. INTRODUCTION

The analysis of eclipsing binary light curves involves several distinct subjects: (1) the physical model; (2) the mathematical and numerical representation; (3) the parameter fit procedure and a discussion of the errors of the estimated parameters. The first subject requires choice of a physical model appropriate to the investigation. The choice is guided by current knowledge of binary star physical properties and is conditioned by due regard for detectability of separate physical parameters, in the environment of current observational technique. The model should be general enough for wide applicability, with only the model parameters varying from case to case. Thus, the Russell model has been superseded by the Roche model, not because the former was incapable of representing observational data extant at the time, but rather because of recognized deficiencies in the physical accuracy of the former as compared with the latter. The Roche model itself is deficient in the sense that it provides no basis for calculating physical properties that depend explicitly on density in the photospheric region. Thus, the Roche model cannot be used to calculate a model atmosphere for a tidally and rotationally distorted star, in which the atmosphere parameters vary with both polar angle and azimuth. On the other hand, it is not clear that an improved model is needed by current observations, generally speaking.

The second subject concerns the mathematical model, specifically an appropriate means of representing the geometry of the system and its physical properties. Major problems in the numerical representation are numerical accuracy and the associated large amount of computer time involved in sophisticated mathematical models.

The third subject concerns a parameter optimization procedure. The direct problem—given model parameters, to calculate the resultant light curve—is analytically straightforward. The inverse problem (IP)—given the light curve and perhaps also the radial velocity curve, to calculate the model parameters—involves a number of hidden hazards. Parameters may be correlated. A parameter which may be well determined for complete eclipses is poorly determined for partial eclipses. The IP is strongly nonlinear, and this feature deserves consideration in any choice among options for parameter optimization. Since an initial selection of parameters may be little better than a guess, use of an optimization procedure is essential. In some cases, prior familiarity with the effect of a particular parameter on a light curve can support a "cut and try" optimization. There is no guarantee that his procedure will produce the best set of parameters. Also, the procedure is unsystematic and can be wasteful of computer time. The error in the complete problem solution involves contributions from several sources. The mathematical procedure which translates the physical model into calculable representation may have inherent limitations. For example, the Roche model radius must be calculated numerically; this feature draws attention to the convergence properties of the algorithm which calculates the radius. Implementation of the mathematical procedure involves inevitable truncation errors in a finite word-length computer. A compromise must be made in the choice of grid density, on the stellar component photospheres, between desirable numerical accuracy and available CPU time. The procedure of fitting the mathematical model to the observed data involves a separate error. When the fitting procedure is complete, as determined by an appropriate criterion, the remaining errors arise from the unavoidable observational noise and possible systematic effects, the latter indicating a defect in the basic physical model. This assumes the observational data are free of extinction reduction errors.

This paper focuses mainly on the third subject and presents an application of the simplex procedure to the optimization of eclipsing binary model parameters. The simplex procedure is a direct search method and therefore does not require partial derivatives of the calculated light with respect to the system

parameters. It has achieved increasing popularity in other fields, for instance in analytical chemistry (Long 1969; Yarbro & Deming 1974).

Section II of this paper discusses properties and advantages of several optimization techniques. Section III considers the implementation of the simplex algorithm for the present application. Illustrative results for a test case, MR Cygni, are in § IV. Future prospects are in § V and conclusions are given in § VI.

II. ECLIPSING BINARY OPTIMIZATION TECHNIQUES

a) *Differential Corrections*

This procedure linearizes the problem by expanding the expression for the weighted sum of squares of system light residuals (WSSR) about the currently-adopted parameters $X^{(0)}$. To the best of our knowledge, no other optimization technique is in current use in light synthesis programs. The expansion adopts a calculation of partial derivatives $\partial l_c^{(i)}/\partial x_j$ of system light $l_c^{(i)}$ with respect to individual parameters x_j in terms of finite symmetric or asymmetric differences [subscript c indicates a calculated quantity, and superscript (i) labels the successive phases of observation]. The Wilson-Devinney (1971, hereafter WD) program and also the Wilson (1979) program, which together are by far the most widely used current optimization programs, use this procedure. A set of normal equations connects the unknown parameter increments with light residuals. Thus, if x_j represents the jth parameter to be adjusted, the program calculates $\partial l_c^{(i)}/\partial x_j$. The light residual at orbital phase $\psi^{(i)}$ is $\delta l^{(i)} = l_0^{(i)} - l_c^{(i)}$, where $l_0^{(i)}$ is the observed light and $l_c^{(i)} = l_c[\psi^{(i)}, X^{(0)}]$, where $X^{(0)}$ designates the current set of parameters. The light residuals permit calculation of WSSR. The least-squares criterion and linearization lead to a linear set of normal equations, of the form $\mathbf{DC} = \mathbf{G}$, where the matrix $\mathbf{D}$ contains all partial derivatives and also all information needed to calculate the covariance matrix for the parameters to be estimated, $\mathbf{C}$ is a one-column matrix of the unknown parameter increments, and $\mathbf{G}$ is a one-column matrix containing weighted residuals (Linnell and Proctor 1972). The matrix $\mathbf{D}$ then is inverted by a standard algorithm to produce $\mathbf{D}^{-1}$, and the solution vector follows from $\mathbf{C} = \mathbf{D}^{-1}\mathbf{G}$. It is the least-squares criterion which makes calculation of $\partial l_c^{(i)}/\partial x_j$ necessary. Due to the linearization it is necessary to iterate the process. It is an implicit expectation that the corrected parameter set migrates to the true minimum of WSSR.

The following problems arise with this procedure.

1. The differentials correction (DC) procedure is a local procedure, i.e., it requires that the initial parameter set $X^{(0)}$ be "close enough" to the local minimum at X^*. What "close enough" means can be said more precisely, as in Osborn (1972), in terms of the local curvature in the hypersurface produced by WSSR = WSSR(l_0, X). These criteria cannot be checked easily. It may happen that the initial approximation $X^{(0)}$ is far from X^*, and the iterative procedure diverges.

2. Convergence to an accurate solution requires accurate values of the derivatives, particularly during the final iterations. The common finite difference approximation is $\partial l_c^{(i)}/\partial x_j \approx \Delta l_c^{(i)}/\Delta x_j$. Formally, an accurate approximation requires a small Δx_j. The smaller Δx_j is made, the more stringent is the requirement for numerical accuracy in calculating $\Delta l_c^{(i)}$, since the latter quantity becomes a smaller and smaller difference between nearly equal numbers. If one needs the partial derivatives of m parameters using asymmetric differences, one has to calculate $m + 1$ light curves, and for symmetric differences, $2m + 1$. Since the calculation of one light curve can consume an appreciable amount of computer time, all DC procedures are expensive in terms of computer time.

3. The DC procedure is sensitive to correlation between parameters; this can lead to failure to achieve an iterative solution.

4. The matrix $\mathbf{D}$ is often nearly degenerate. The problem becomes particularly ill conditioned when a large number of parameters are included simultaneously. Wilson and Biermann (1976) introduced the "method of multiple subsets." This is also used by Khaliullina and Khaliullin (1984). There do not appear to be any quantitative mathematical theorems about the requirements of this technique, and it difficult to give a systematic algorithm to solve a problem by applying this method.

5. Criteria to decide when to terminate DC iterations can be arbitrary. Choosing to stop when the correction to any parameter is $\sim 1/10$ its calculated probable error does not guarantee that an asymptotic convergence is complete. Since the probable errors depend on WSSR, a situation could arise in which a calculated correction to a parameter is small, at a particular iteration, but in fact the solution is not close to the true WSSR minimum. Even worse, it may happen that the WSSR increases after a correction. As one notices in Wolfschmidt (1980), for some stars the successive iterations may be erratic and may require up to 250 steps of iteration for convergence.

b) *Alternatives to the Differentials Correction Method*

The problems just discussed can be reduced to problems in terms of accuracy and amount of CPU time required to compute the partial derivatives. Two improvements are possible. First, one could seek to calculate partial derivatives analytically. This approach, introduced by Linnell and Proctor (1972) in Russell Model solutions, achieves high accuracy and avoids the large number of calculated light curves otherwise required. This method is not possible for all parameters; a separate discussion will appear in a following paper by one of us (J. K.). Second, one could try to avoid the use of partial derivatives at all. That would require a different optimization procedure. There are other functions than WSSR that may be chosen for optimization, and there are other optimization procedures than the DC in use by astronomers. Examples are the procedures described in "maximum entropy methods" (Skilling and Bryan 1984). Horne (1985) applied these methods to map accretion disks in binary star systems. Unfortunately these methods require calculation of derivatives. Overviews of "direct search methods"—which by definition do not require derivatives at all—are given in Murray (1972). Compared with other gradient-free methods, for instance the Fibonacci line search, the simplex algorithm as given by Nelder and Mead (1965, hereafter NM) becomes more efficient when there are more than two or three parameters to be adjusted. We stress that the efficiency of a procedure is a matter of the efficiency of the convergence effectiveness *and* the efficiency of calculation of all quantities needed in the algorithm. DC and all derivative dependent procedures are most efficient with respect to convergence effectiveness but are very inefficient when one considers the need to calculate $m + 1$ or $2m + 1$ light curves at each iteration.

c) *The Simplex Algorithm*

As a direct-search method, the simplex algorithm (SA) does not depend on gradients (first-order derivatives) or quadratic

forms (second-order derivatives). Instead, it compares function values at the $m + 1$ vertices of a geometrical figure called a simplex. In a first realization by Spendley, Hext, and Himsworth (1962), this figure moved through parameter space by means of an operation which later was called "reflection." In this form it is similar to a "steepest descent" procedure. However, the maximum gradient direction of WSSR was not calculated explicitly. In the much more efficient form by NM, four operations on the simplex (see Appendix) are allowed. By means of "reflection," "expansion," "contraction," and "shrinkage" the simplex moves through parameter space, adapts itself to the local topology, and contracts to the final minimum. These operations can be controlled by three or four parameters. The simplex algorithm can easily be understood in geometrical terms. There are $m + 1$ different parameter sets X with m parameters each, and WSSR(X) is defined to be the $(m + 1)$th entity for an $m + 1$ dimensional simplex. To find the local minimum on the WSSR(X) hypersurface, the simplex determines, at each step, the vertex with highest and lowest WSSR value and eliminates the one of highest WSSR value by the scheme given in the Appendix. The simplex can never lose the best value (the WSSR of smallest value). The speed of convergence usually is reduced when there are parameter correlations (valleys in the WSSR landscape). In our experience the simplex does not fail even in these cases. The speed of convergence is only slightly dependent on the initial simplex, calculated according to Yarbro and Deming (1974, hereafter YD). If necessary, the simplex can have an arbitrarily large volume in parameter space. Therefore it is a global procedure for "searching" through parameter space. The calculation of the initial simplex requires the calculation of $m + 1$ light curves for adjusting m parameters. Later the algorithm does not depend on m directly. By contrast, as mentioned earlier, the DC and similar procedures have the disadvantage that they can only improve an estimated or current iterative parameter solution, but may diverge if the current parameter set is far from the correct set. Using the final simplex, it is possible to calculate the covariance matrix of all estimated parameters and use this to estimate their errors.

III. PROGRAM TO IMPLEMENT SIMPLEX METHOD

A global control program called LCCTRL—written in VAX 750 FORTRAN—provides the user with all I/O routines which are needed, prepares input and output data for graphical representation, allows separate calls for generation of light curves, and creates connections between the simplex algorithm and the WD program. We initially implemented the simplex method with the WD program because it is simpler than the Linnell (1984) program. (We have now implemented the simplex method with the Linnell program.) A subsequent paper will compare the WD and Linnell program results for MR Cygni. LCCTRL allows automatic iteration with both the DC and the SA. Automatic iteration is regarded as dangerous with the DC, but LCCTRL has a set of constraints and other checks to avoid unphysical parameters as arguments for WD. Furthermore, LCCTRL can be used to apply a sensitivity analysis to evaluate the errors of the parameters, as suggested by Caceci and Cacheris (1984), different from the covariance matrix method. The physical model used to generate light curves is that of WD with modifications up to 1978. Some subroutines have been changed, and different structures have been substituted in parts of the original program. The program uses only integer, double precision, and character variables and uses the new structural features of FORTRAN 77 wherever possible. The algorithm given by YD to produce the initial simplex has been changed. The user can now chose to have the initial parameter set $X^{(0)}$ in the sense of YD, or to define an initial simplex with a center near the $X^{(0)}$, or to give a hypercube on whose edges lie the vertices of the initial simplex. LCCTRL provides the user with some "restart" features. This allows one to check the current iteration while the program is running, to control it interactively, and to restart the program if it has been stopped. Subroutine SIMPLEX occupies 350 program statement lines, including comment lines. We use a criterion suggested by NM to stop iterations. The criterion compares the standard deviation of residuals with a previous value derived from the observed light curve. After 20 or 30 iterative steps (with 320 grid points on each star, 168 data points, and nine free parameters and requiring roughly 120 or 150 minutes VAX CPU time) we achieved an accuracy of 0.02 for the standard deviation of the MR Cygni residuals. We terminated iterations after 60 steps and achieved a standard deviation of 0.007. By comparison, one step applying subroutine DC (differential correction) to 336 data points and nine adjustable parameters and 2000 grid points on each star required $\sim$75 minutes CPU time on the same machine.

IV. RESULTS FOR MR CYGNI

We chose to study MR Cygni for a variety of reasons. The mass ratio is still uncertain. WD used the photometric data of Hall and Hardie (1969) and assumed a mass ratio of 0.83. They (WD) did not adjust this quantity. Hill and Hutchings (1972) prefer a value of 0.55 derived from their spectroscopy. WD fixed the polar temperature of the second star to a fixed value of 13,500 K. This polar temperature is not appropriate for a B8 secondary, the spectral type derived by Hall and Hardie (1969) from color data. We used the Hall and Hardie (1969) UBV data. The efficiency of SA allowed us to use all 336 data points. In practice, DC typically has used about 50–70 normal points. We stress that the great efficiency in SA computer time arises mainly from the fact that we do not depend on a very high precision grid. The only requirement was that the error produced by the grid was smaller than the residuals $l_0^{(i)} - l_c^{(i)}$. In the final stage of the fit, residual errors were of order 10^{-3}. Increasing the number of grid points from $\sim$300 to $\sim$2000 produced changes in computed light values $l_c^{(i)}$ on the order of 10^{-4}. It proved useful to select a subset of the individual observations during initial steps of the simplex. The complete set of individual observations was used in the final adjustment.

Figures 1 and 2 illustrate the simplex operation with the MR Cyg V data. For this demonstration purpose we adjusted only the mass ratio and orbital inclination; we fixed all other parameters at their final values achieved in the separate complete iteration procedure. A limit of two parameters enables a presentation of simplex operations on a plane. The initial range of parameters leads to a triangular simplex with one vertex, marked 0, at $q = 0.3$, $i = 75°$ in Figure 1. WSSR is highest for vertex 0. The next simplex is produced by reflection with respect to the center of all other vertices. It was accepted, since the new vertex had a lower WSSR. At this stage—not displayed in this diagram—the SA made an unsuccessful trial to extend the reflection. "Reflection" is designated by an operator $\hat{\alpha}$ on the diagram. Simplex 2 is produced by "contraction," controlled by an operator $\hat{\beta}$. Figure 2 corresponds to the dashed box in Figure 1. The final simplex in Figure 1 has heavy lines and vertices 4, 6, and 7. The same simplex vertex 7

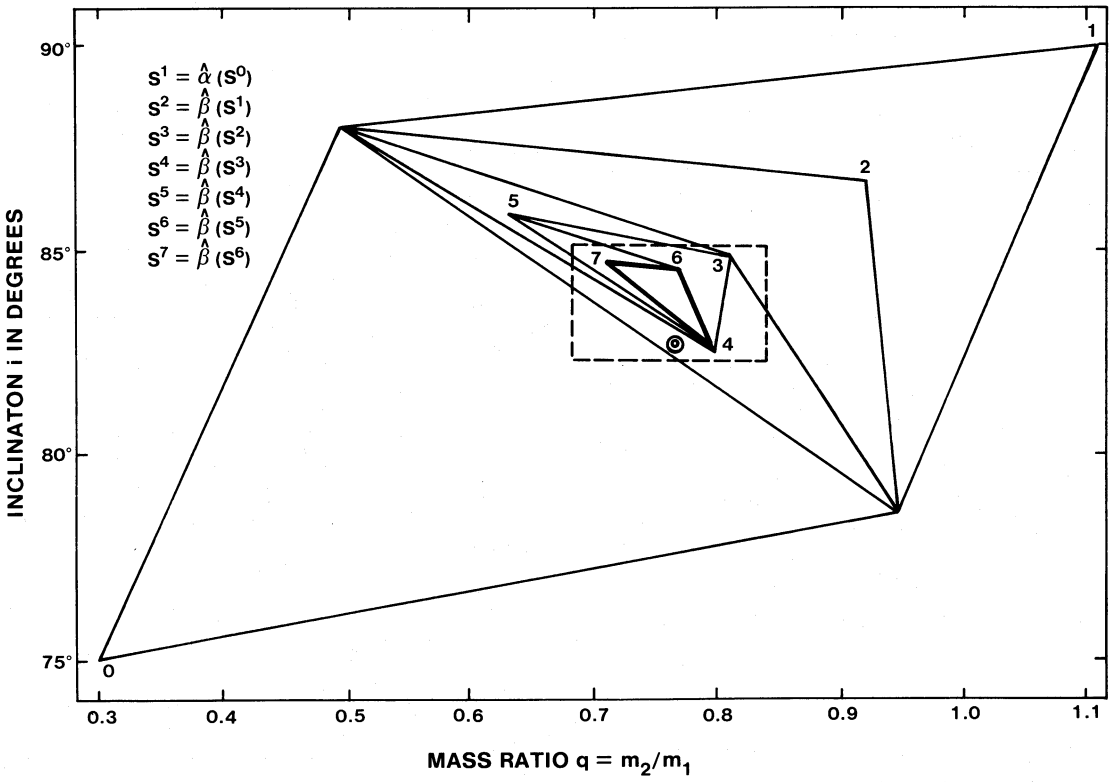

FIG. 1.—Change of simplex shape while moving through *i-q* parameter plane

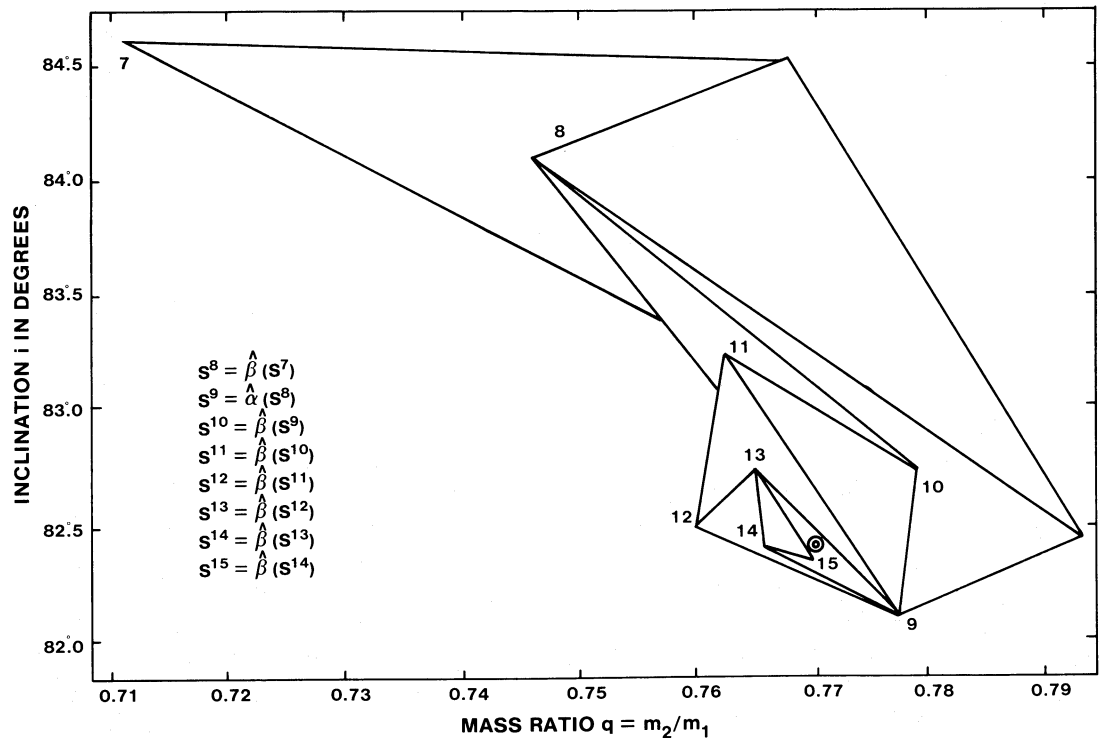

FIG. 2.—Change of simplex shape while moving toward local minimum at $i = 82.^{\circ}43$, $q = 0.770$

appears in Figure 2. Not all sides of the simplex triangles have been displayed in Figure 2, for clarity of presentation. Figure 2 illustrates the simplex migration concluding with step 15. The small double circle shows the vertex location on step (or iteration number) 25. Figure 3 shows the decreasing ranges of the two parameters. Each vertical bar, for a given step or iteration number, shows the range of that parameter for the current simplex. Figure 4 shows the largest and smallest standard deviation for the current simplex, as a function of step number. Note that although the vertical bar length may increase in some instances with a successive step, the top of the bar, representing the maximum standard deviation for the simplex, never moves higher. Neither does the smallest standard deviation, the bottom of each bar.

It may be appropriate to start with analysis of individual wavelength light curves. The final geometric quantities should have values independent of wavelength. A large disagreement among different wavelength solutions is indicative of a failure of the underlying physical model, assuming correct mathematical and numerical models. If the solutions fit together well, one can attempt a simultaneous fit, as suggested by Wilson and Devinney (1972). The newer version of WD allows the user to choose a mode between -1 and 6. The 1971 version of WD ran only in mode 0. We used the same mode to provide a valid comparison, even though that mode is not the most sensible one. L_1, L_2, T_1, and T_2 are not coupled in this mode as they should be, physically. An exact direct comparison was difficult, since the program version we were using had undergone some changes from the 1971 version. Our version includes an improved treatment of the reflection effect, as given by Wilson *et al.* (1972). For simplicity in this initial test, we have set all weights to unity. The separate fits to the V, B, and U data are in Figures 5, 6, and 7 respectively. The parameters subject to adjustment in these fits were: i, T_2, Ω_1, Ω_2, q, L_1, L_2, x_1, and x_2. WD, as controlled by LCCTRL, provides a routine to calculate the parameter errors. However, as suggested by NM, the errors could have been calculated from the final simplex.

A plot of residuals appears beneath each separate wavelength light curve fit. The V residuals show no systematic trend with phase, and no seriously discrepant observations are apparent. There are one seriously discrepant B observation and two fairly discrepant U observations. A 3 σ test would eliminate these points and produce improved standard deviations. It is our contention that light curve solutions should generally show a plot of residuals of all data points, particularly when the plotted light curve appears to fit the observations accurately. This was not possible in the WD solution, since normal points were used. Note that the B and U residuals generally show no systematic trends except within secondary minimum. The opposite sense of residuals tilts suggests a possible problem with extinction correction.

To effect a comparison with the WD solution we have calculated residuals, using the WD program and the parameters listed for their V curve solution. Unfortunately, WD do not give the unnormalized luminosities L_1 and L_2 for the B curve. The V residuals for the original WD solution are in Figure 8. A slight systematic trend is evident in the V residuals, with the residuals near phase 0.50 slightly above the residuals near

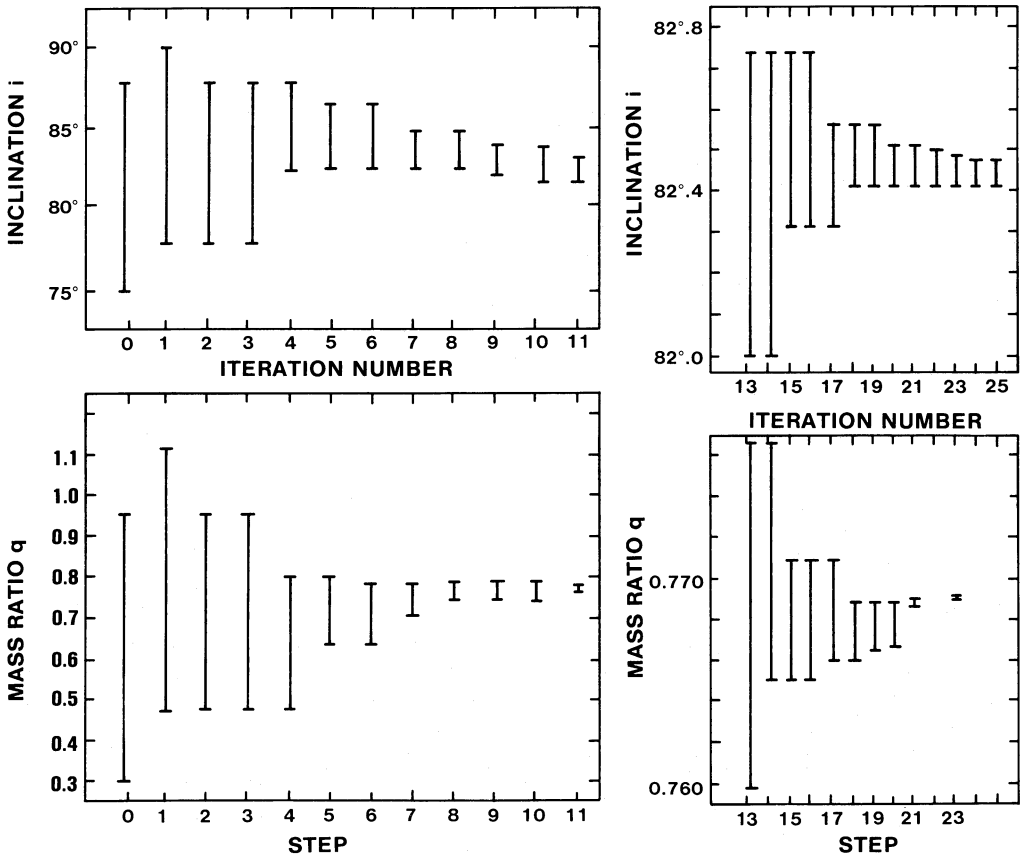

FIG. 3.—Maximum and minimum values of i and q, within simplex, as function of step or iteration number

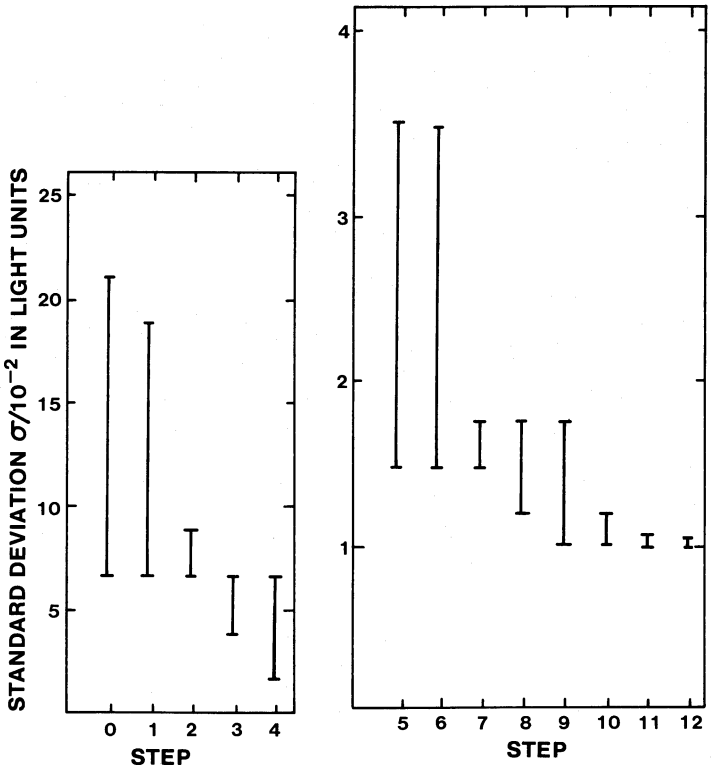

FIG. 4.—Standard deviation of V residuals as function of step number

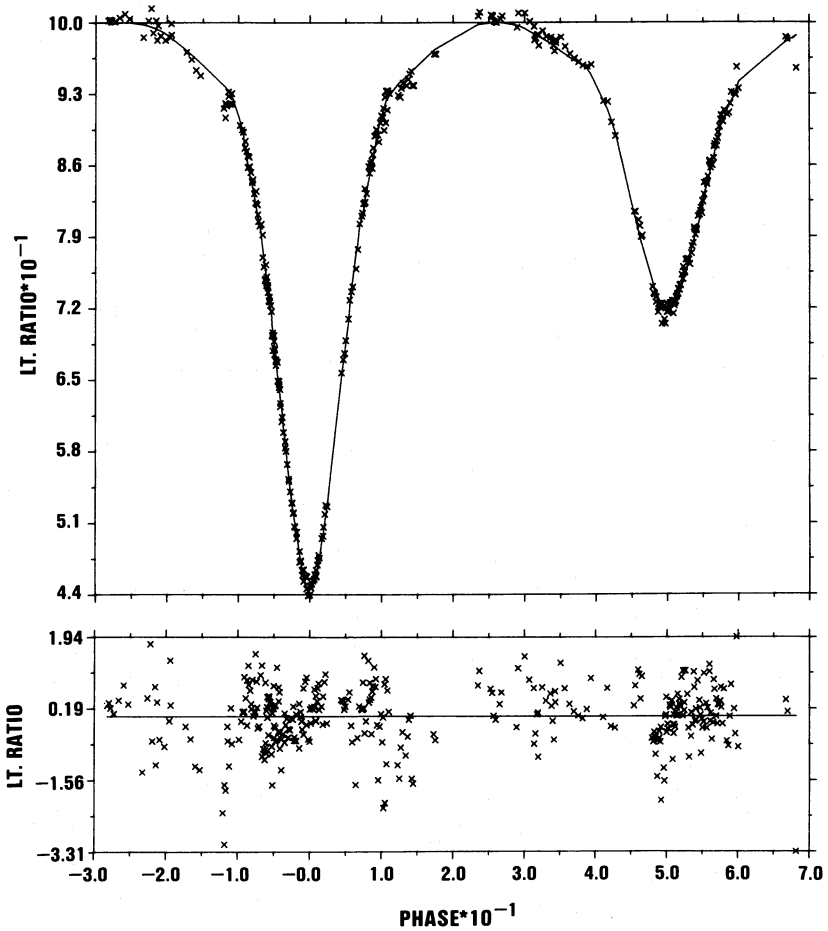

FIG. 5.—Fit to V data, WD program. *Bottom*, Light residuals times 10^2.

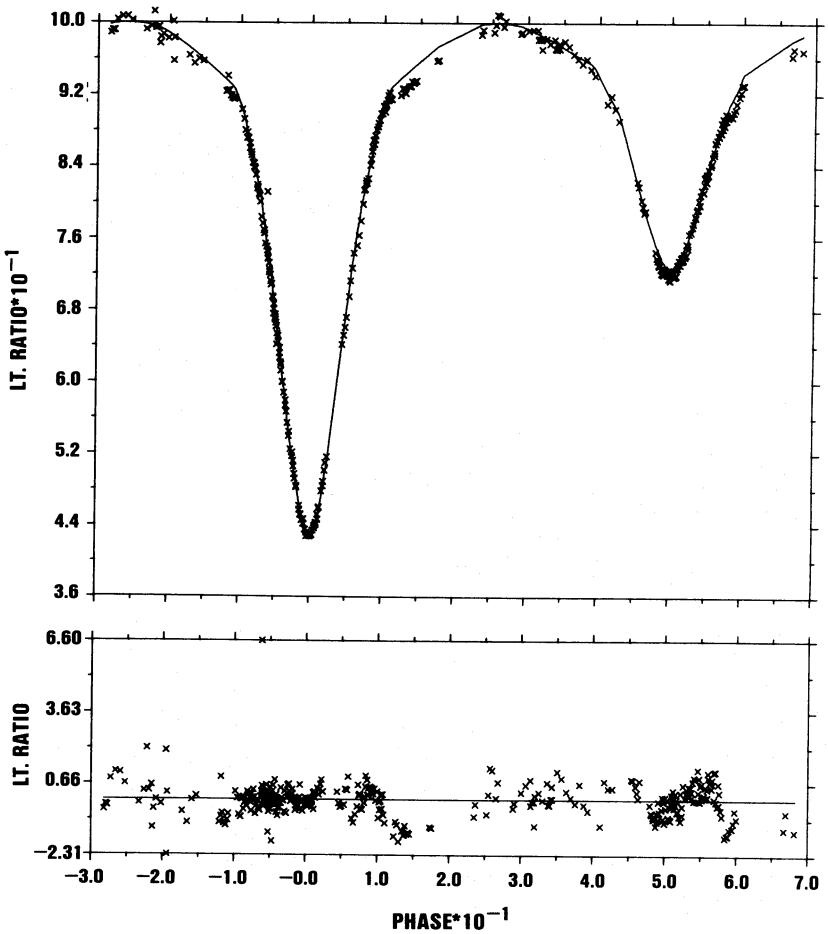

FIG. 6.—Fit to *B* data, WD program. *Bottom*, Light residuals times 10^2.

phase 0.0. The residuals are mainly negative, indicating that either the model was incapable of representing the observational data or that the optimization process—particularly for the inclination—had not gone to completion.

Parameters of the system, separately in U, B, and V, are in Table 1. For convenience we list the same parameters from the separate WD B and V solutions. Here, L_1 and L_2 are the component luminosities in the units defined in the WD program; l_1 and l_2 are the corresponding fractional system luminosities. Other parameters have their usual meanings (Wilson and Devinney 1971). Note that T_1 must be adopted from considerations external to the light synthesis programs. Values from theory have been adopted for g_1, g_2, A_1, and A_2. We list formal probable errors only for quantities subject to direct adjustment. Note their substantial reduction from the original WD solution.

A comparison of the geometric parameters in the separate light curve solutions led us to attempt a simultaneous fit to the B and V curves. The presence of the Balmer discontinuity in the U band commended exclusion of the U curve at this stage. We included the U curve in a subsequent simultaneous solution, for test purposes. We chose mode 2 as more appropriate, physically, than mode 0. In this case the unnormalized luminosities $L_{1,2}$ and the polar temperatures $T_{1,2}$ are coupled by the Planck function. This optimization employed light-level–dependent weights, with a light-level exponent of 1 ($b = 1$ in the nomenclature of Linnell and Proctor 1970). An important point for this paper is that the simplex procedure produced an excellent simultaneous solution also. The limb-darkening coefficients were allowed to vary over a quite large range [0.1, 0.6]. The final values were in better agreement with Kurucz atmosphere values than was true in the original WD solution. We used linear limb-darkening tabulations by Wade and Rucinski (1985) for this comparison. Our trials covered a large q range [0.3, 1.0]. Our solutions never converged to a mass ratio less than 0.70. Table 2 provides a summary of system parameters from our simultaneous V, B solution. The first solution (col. [2]) provides final results in mode 2 using the parameters listed by Wilson and Devinney (1972). Theirs was a simultaneous B, V solution, indicated by BV in the column heading. The wavelength-dependent parameters are in the sequence B, followed by V in parentheses. The second solution (col. [3]), in mode 2, used the simplex algorithm for optimization. The third solution (col. [4]) with light synthesis by SYNPGM, lists L_1 and L_2 bolometric luminosity in solar luminosity units. The values of l_1 and l_2 are the corresponding bolometric fractional luminosities. SYNPGM refers to the Linnell program. We do not list probable errors for the SYNPGM solution, since we have not yet implemented means to calculate them with the simplex procedure. The simultaneous U, B, V solution was less satisfactory in the sense that a larger WSSR remained after many iterations. We attribute this behavior to a model failure arising from the presence of the Balmer discontinuity, together with the attempt by the model to represent the continuous

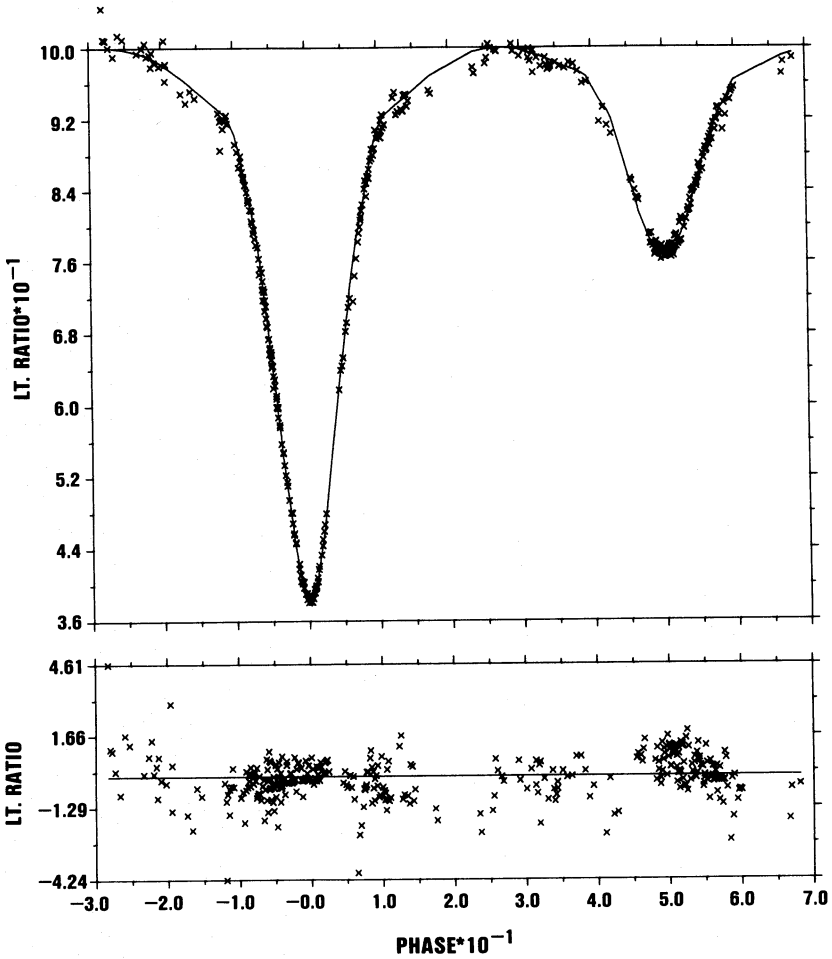

FIG. 7.—Fit to *U* data, WD program. *Bottom*, Light residuals times 10^2.

radiation curve by a blackbody. We defer further discussion of a physical model of MR Cygni to a subsequent paper.

V. FUTURE PROSPECTS

The SA version we are now using could be improved in several ways. First, some sophisticated enhancements of the SA itself, discussed by Parkinson and Hutchinson (1972), are possible. Second, a more efficient formulation of SA is possible specifically for eclipsing binary light curve applications. Next, a combination of the SA as the initial search procedure and a following procedure, which makes use of the information that one is close to the local minimum, may be possible. In the current version LCCTRL already provides an option to switch from SA to DC. This feature was not used, since we wished to avoid the calculation of derivatives.

VI. CONCLUSIONS

1. We have demonstrated the utility of a new parameter optimization technique for binary star light curve solutions. This technique, the simplex algorithm, cannot diverge, does not require partial derivatives of calculated light with respect to system parameters, is capable of automatic iteration, is operationally stable, and is computationally efficient.

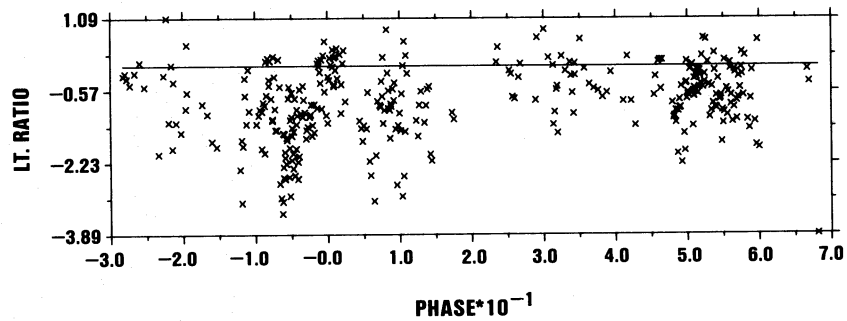

FIG. 8.—*V* residuals from original WD solution

TABLE 1

SEPARATE *UBV* LIGHT CURVE ANALYSES

PARAMETER	WILSON AND DEVINNEY 1971		WD (mode 0) + SIMPLEX		
	B	*V*	*U*	*B*	*V*
i	82°.29 ± 0°.30	83°.28 ± 0°.61	82°.52 ± 0°.03	81°.79 ± 0°.12	82°.81 ± 0°.04
λ (nm)	435	550	365	435	550
L_1	...	9.8416 ± 0.1016	9.8078 ± 0.0574	9.2258 ± 0.0595	9.6337 ± 0.0461
L_2	...	2.6984 ± 0.0953	2.2831 ± 0.0431	2.9311 ± 0.0487	2.8514 ± 0.0400
l_1	0.7698 ± 0.0087	0.7848 ± 0.0081	0.8110 ± 0.0047	0.7589 ± 0.0049	0.7716 ± 0.0037
l_2	0.2302 ± 0.0080	0.2152 ± 0.0076	0.1890 ± 0.0036	0.2411 ± 0.0040	0.2284 ± 0.0032
x_1	0.75 ± 0.06	0.62 ± 0.08	0.54 ± 0.04	0.59 ± 0.05	0.414 ± 0.038
x_2	−0.36 ± 0.21	−0.11 ± 0.12	0.59 ± 0.22	0.49 ± 0.21	0.637 ± 0.160
g_1,[a] A_1[a]	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
g_2,[a] A_2[a]	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
T_1[a] (K)	18,000	18,000	18,000	18,000	18,000
T_2 (K)	13,500[a]	13,500[a]	13,039 ± 595	15,745 ± 688	15,811 ± 528
q	0.83[a]	0.83[a]	0.740 ± 0.004	0.824 ± 0.005	0.767 ± 0.004
Ω_1	3.828 ± 0.017	3.749 ± 0.017	3.796 ± 0.002	3.884 ± 0.005	3.671 ± 0.005
Ω_2	3.906 ± 0.030	4.009 ± 0.035	3.551 ± 0.002	3.762 ± 0.001	3.674 ± 0.002
r_1 pole	0.329	0.337	0.323	0.327	0.340
r_1 point	0.371	0.388	0.356	0.366	0.385
r_1 side	0.340	0.350	0.342	0.348	0.363
r_1 back	0.356	0.369	0.346	0.353	0.369
r_2 pole	0.290	0.280	0.300	0.303	0.295
r_2 point	0.322	0.307	0.344	0.343	0.333
r_2 side	0.298	0.287	0.308	0.311	0.303
r_2 back	0.312	0.299	0.329	0.330	0.321
Normal points	~50	~50	...	...	...
Data points	...	...	336	336	336
$\sigma(O-C)$	...	0.012	0.008	0.007	0.008

[a] Fixed quantities.

TABLE 2

SIMULTANEOUS *B*, *V* SOLUTION WITH WD and LINNELL PROGRAMS

Parameter (1)	Wilson and Devinney 1972 *BV*, *B*(*V*) (2)	Wilson + Simplex *BV*[mode 2], *B*(*V*) (3)	SYNPGM + Simplex *B* and *V* curve (4)
i	82°.89 ± 0°.23	83°.03 ± 0°.03	82°.50
L_1	... ...	9.3610(9.2988) ± 0.0236(0.0327)	1830 $L_\odot$
L_2	... ...	2.8215(3.2352) ± 0.0071(0.0114)	340 $L_\odot$
l_1	0.7950(0.7785) ± 0.0059(0.0059)	0.7684(0.7419) ± 0.0019(0.0026)	0.8431
l_2	0.2050(0.2215) ± 0.0054(0.0055)	0.2316(0.2581) ± 0.0006(0.0009)	0.1569
x_1	0.70 (0.65) ± 0.03 (0.04)	0.52 (0.59) ± 0.15 (0.02)	Grid values
x_2	0.00 (0.00) ± 0.00 (0.00)	0.14 (0.27) ± 0.18 (0.02)	Adopted from Kurucz atmospheres
g_1[a]	1.00	1.00	1.00
g_2[a]	1.00	1.00	1.00
A_1[a]	1.00	1.00	1.00
A_2	1.00	0.53 ± 0.04	0.53
λ (nm)	435 (550)	435 (550)	435 and 550
T_1[a] (K)	18,000	18,700	18,700
T_2 (K)	13,500[a]	12,934	12,808
q	0.83[a]	0.702 ± 0.002	0.788
Ω_1	3.776 ± 0.013	3.611 ± 0.004	3.841
Ω_2	3.990 ± 0.022	3.482 ± 0.003	3.701
r_1 pole	0.335	0.340	0.324
r_1 point	0.382	0.381	0.360
r_1 side	0.347	0.361	0.337
r_1 back	0.364	0.367	0.348
r_2 pole	0.282	0.296	0.299
r_2 point	0.310	0.339	0.341
r_2 side	0.289	0.318	0.313
r_2 back	0.302	0.324	0.326
σ	...	0.0067(0.0088)[b]	0.013(0.015)[b]
Data points	...	2 × 335	2[a] 335

[a] Fixed quantity.
[b] Weighted ($b = 1$).

2. Separate V, B, and U solutions of MR Cygni data achieved reasonably accordant geometric parameters. The solution used individual observations rather than normal points.

3. We find limb-darkening coefficients that agree better with values from Kurucz atmospheres than was true for the WD solution. The original WD solution, using the initial version of the WD program, was troubled by negative limb-darkening coefficients for the secondary component.

4. We provide separate residuals plots in V, B, and U as a standard feature of light curve solutions. The plots generally show no systematic trends and have individual means of zero. As a separate test we calculated residuals, in mode 0, using the WD solution parameters and the V light curve. A slight systematic trend was evident, and the mean was not zero, indicating the possibility that the solution had not gone to completion.

5. The simplex procedure worked well in achieving a simultaneous solution with the combined V and B data, and again with the combined U, B, and V data. The latter solution was less satisfactory than the former. We attribute this feature to failure of the blackbody approximation together with presence of the Balmer discontinuity in the U band. We defer detailed physical discussion of the system to a subsequent paper.

J. K. wants to thank R. E. Wilson for his helpful discussions of the WD program during a Florida visit and gratefully acknowledges financial support from a stipendium of the Cusanuswerk (Federal Republic of Germany).

APPENDIX

To adjust m parameters $(x_1, \ldots, x_m) = \boldsymbol{x}$, such that a real function $f(\boldsymbol{x})$ is minimized, define a "simplex" in an m-dimensional space as a set of $m + 1$ points which span a subspace. Let $V_i^{(k)}$, $i = 1, \ldots, m + 1$ be the vertices of the simplex $S^{(k)}$ in the kth step of iteration with coordinates $\boldsymbol{x}_i^{(k)}$. For practical purposes choose $S^{(k)}$ as an $(m + 1) \cdot (m + 1)$ dimensional matrix $S_{ij}^{(k)}$ defined by

$$S_{ij}^{(k)} = \begin{cases} x_{ij}^{(k)}, & 1 \le j \le m, \\ f_i = f(\boldsymbol{x}_i), & j = m + 1. \end{cases} \tag{A1}$$

Further definitions hold for each step separately

$$f_h = \max\,[f_i,\ 1 \le i \le m + 1], \tag{A2}$$

$$f_s = \max\,[f_i,\ 1 \le i \le m + 1,\ i \ne h], \tag{A3}$$

$$f_l = \min\,[f_i,\ 1 \le i \le m + 1], \tag{A4}$$

$$\boldsymbol{x}_c = \frac{1}{m} \sum_{\substack{i=1 \\ i \ne h}}^{m+1} \boldsymbol{x}_i\ ; \quad \text{center of simplex}. \tag{A5}$$

The initial simplex $S^{(0)}$ depends on m, an initial vector $\boldsymbol{x}_1^{(0)}$ and a vector $\Delta \boldsymbol{S} = (S_1, \ldots, S_m)$ which contains a step size for each parameter. An algorithm by Spendley, Hext, and Himsworth (1962) and YD constructs $S^{(0)} = S^0(m, \boldsymbol{x}_1^{(0)}, \Delta s)$ using

$$p = [(m + 1)^{1/2} + m - 1]/(\sqrt{2}m), \tag{A6}$$

$$q = [(m + 1)^{1/2} - 1]/(\sqrt{2}m), \tag{A7}$$

$$x_{ij}^{(0)} = x_{1j}^{(0)} + s_j \cdot \begin{cases} 0, & i = 1, \\ p, & j = i - 1, \\ q, & \text{otherwise}. \end{cases} \tag{A8}$$

We also use $S^{(0)} = S^{(0)}[m, \boldsymbol{x}_1^{(0)} - q\Delta \boldsymbol{S}, \Delta \boldsymbol{S}]$ and $S^{(0)} = S^{(0)}[m, \boldsymbol{x}_1^{(0)}, \Delta \boldsymbol{S}/p]$ as alternative procedures for the initial simplex $S^{(0)}$.

In agreement with the nomenclature used in Figure 9, define the vector functions

$$\boldsymbol{x}_\alpha = \boldsymbol{x}_\alpha(\boldsymbol{x}) := (1 + \alpha)\boldsymbol{x}_c - \alpha \boldsymbol{x}\ ; \quad 0 < \alpha = \text{reflection coefficient}, \tag{A9}$$

$$\boldsymbol{x}_\beta = \boldsymbol{x}_\beta(\boldsymbol{x}) := (1 - \beta)\boldsymbol{x}_c + \beta \boldsymbol{x}\ ; \quad 0 < \beta = \text{contraction coefficient} < 1, \tag{A10}$$

$$\boldsymbol{x}_\gamma = \boldsymbol{x}_\gamma(\boldsymbol{x}) := (1 - \gamma)\boldsymbol{x}_c + \gamma \boldsymbol{x}\ ; \quad 0 < \gamma = \text{expansion coefficient}, \tag{A11}$$

$$\boldsymbol{x}_\delta = \boldsymbol{x}_\delta(\boldsymbol{x}) := \boldsymbol{x}_l + \delta(\boldsymbol{x} - \boldsymbol{x}_l)\ ; \quad 0 < \delta = \text{shrinkage coefficient}. \tag{A12}$$

There are four possible operations $\hat{\alpha}$, $\hat{\gamma}$, $\hat{\beta}$ and $\hat{\delta}$ to map $S^{(k)}$ on $S^{(k+1)}$:

1. Reflection $S^{(k+1)} = \hat{\alpha} S^{(k)}$, i.e., $\boldsymbol{x}_h \to \boldsymbol{x}_\alpha(\boldsymbol{x}_h)$, if $f_s > f_\alpha > f_l$;
2. Expansion $S^{(k+1)} = \hat{\gamma} S^{(k)}$, i.e., $\boldsymbol{x}_h \to \boldsymbol{x}_\gamma(\boldsymbol{x}_h)$, if $f_\gamma < f_\alpha < f_l$;
3. Contraction $S^{(k+1)} = \hat{\beta} S^{(k)}$, i.e., $\boldsymbol{x}_h \to \begin{cases} \boldsymbol{x}_\beta(\boldsymbol{x}_h), & \text{if } f_h < f_\alpha \\ \boldsymbol{x}_\beta[\boldsymbol{x}_\alpha(\boldsymbol{x}_h)], & \text{if } f_\alpha < f_h \end{cases}$;
4. Shrinkage $S^{(k+1)} = \hat{\delta} S^{(k)}$, i.e., $\boldsymbol{x}_i \to \boldsymbol{x}_\delta(\boldsymbol{x}_i)$, if $f_\beta \ge f_h$.

These operations are combined as illustrated in Figure 9. In conformity with the results in Parkinson and Hutchinson (1972), we

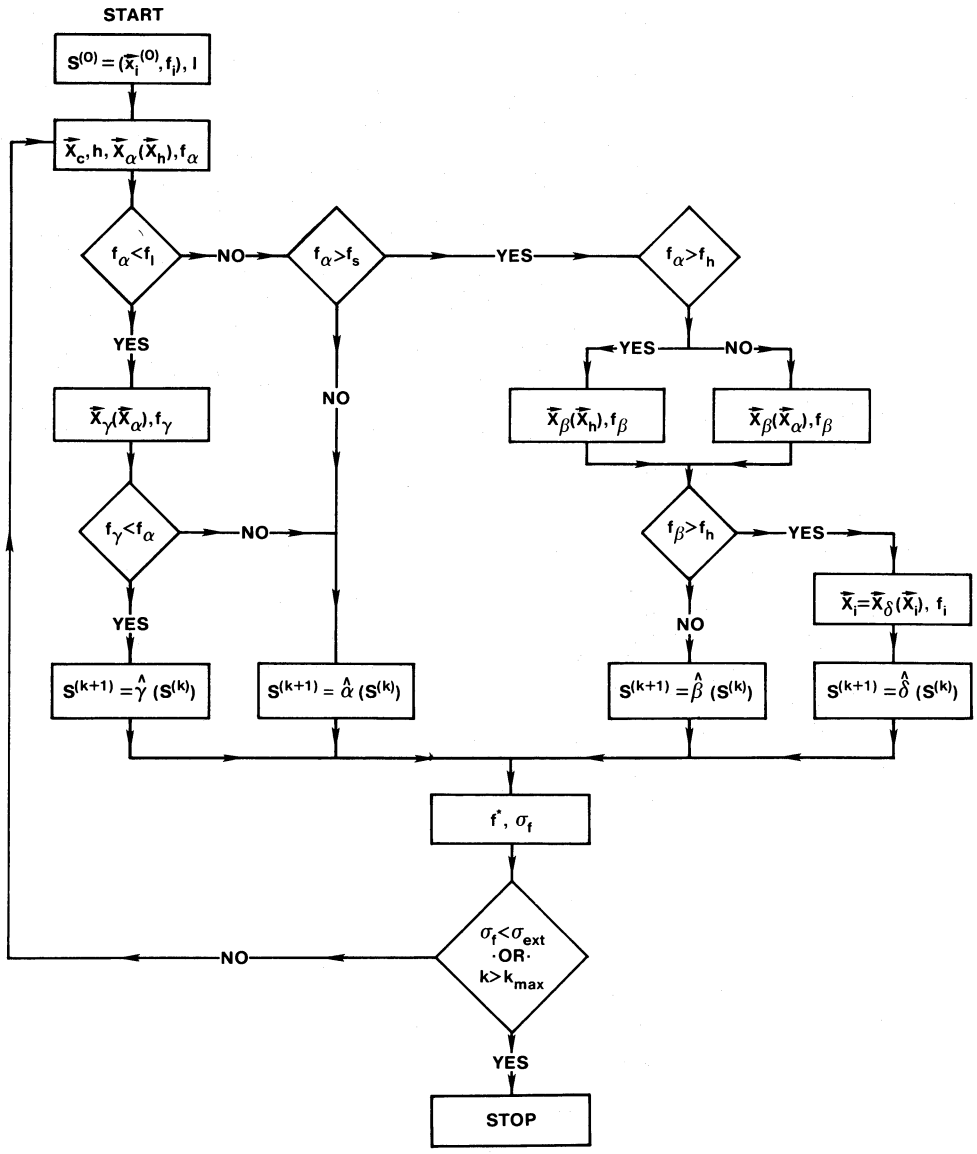

FIG. 9.—Flow chart for simplex algorithm

have chosen $\alpha = 1.0$, $\beta = 0.35$, $\gamma = 2.0$, and $\delta = 0.5$. The iteration was stopped either when $k > k_{max}$, where k_{max} is a previously given maximum number of iterations, or when $\sigma_f < \sigma_{ext}$. Here σ_{ext} is a preset value and usually is related to the standard deviation of the observational residuals and σ_f is the standard deviation of the f_i's in the current simplex:

$$\sigma_f^2 = \frac{1}{m}\sum_{i=1}^{m+1}(f_i - f^*)^2; \quad f^* = \frac{1}{m+1}\sum_{i=1}^{m+1} f_i \,. \tag{A13}$$

A separate criterion would be to stop when $f^* \lesssim \sigma_{ext}$.

REFERENCES

Caceci, M. S., and Cacheris, W. P. 1984, *Byte*, May, 340.
Hall, D. A., and Hardie, R. H. 1969, *Pub. A.S.P.*, **81**, 754.
Hill, G., and Hutchings, J. B. 1972, *Astr. Ap.*, **23**, 357.
Horne, K. 1985, *M.N.R.A.S.*, **213**, 129.
Khaliullina, A. I., and Khaliullin, Kh. F. 1984, *Soviet Astr.*, Vol. **28**, Part 2, p. 228.
Linnell, A. P. 1984, *Ap. J. Suppl.*, **54**, 17.
Linnell, A. P., and Proctor, D. D. 1970, *Ap. J.*, **162**, 162.
———. 1972, *Ap. J. Suppl.*, **24**, 449.
Long, D. E. 1969, *Analytica Chim. Acta*, **46**, 193.
Murray, W. 1972, *Numerical Methods for Unconstrained Optimization* (London and New York: Academic).
Nelder, J. A., and Mead, R. 1965, *Comput J.*, **7**, 308 (NM).
Osborn, M. R. 1972, in *Numerical Methods for Non-Linear Optimization*, ed. F. A. Lootsma (New York: Academic), p. 171.
Parkinson, J. M., and Hutchinson, D. 1972, in *Numerical Methods for Non-Linear Optimization*, F. A. Lootsma, ed. (New York: Academic), p. 115.
Skilling, J., and Bryan, R. K. 1984, *M.N.R.A.S.*, **211**, 111.
Spendley, W., Hext, G. R., and Himsworth, F. R. 1962, *Technometrics*, **4**, 441.
Wade, R. A., and Rucinski, S. M. 1985, *Astr. Ap. Suppl.*, **60**, 471.

Wilson, R. E. 1979, *Ap. J.*, **234**, 1054.
Wilson, R. E., and Biermann, P. 1976, *Astr. Ap.*, **48**, 349.
Wilson, R. E., DeLuccia, M. R., Johnson, K., and Mango, S. A. 1972, *Ap. J.*, **177**, 191.
Wilson, R. E., and Devinney, E. J. 1971, *Ap. J.*, **166**, 605 (WD).
———. 1972, *Ap. J.*, **171**, 413.
Wolfschmidt, G. 1980, Ph.D. thesis, Erlangen-Nuernberg.
Yarbro, L. A., and Deming, S. N. 1974, *Analytica Chim. Acta*, **73**, 391 (YD).

JOSEF KALLRATH: Sternwarte der Universität Bonn, Auf dem Hügel 71, D-5300 Bonn 1, Federal Republic of Germany

ALBERT P. LINNELL: Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824-1116