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Integration theory for the elliptic restricted three-body problem

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Summary. The restricted elliptic three body problem is investigated analytically with respect to the problem of finding limits of space, in which the infinitesimal body, under given circular initial condition, can move. In synodic, rotating and barycentric coordinates we find approximate integrals which limit the region of non-negative velocities. The analytic results are in good agreement with numerical experiments up to eccentricities about $e \approx 0.25$. We also discuss the dependence of these results with respect to the mass ratio μ of the primaries and the initial angle φ.

Key words: elliptic restricted problem – critical orbits

1. Introduction

The restricted three body problem was a subject of many investigations seeking for the boundaries of the region of space in which the infinitesimal body, under given initial conditions, can move. Particularly in the circular restricted three body problem [CP] such works (Szebehely, 1980; Szebehely and McKenzie, 1981) make use of the Jacobi integral. Extensions of this approach are discussed in Ovenden and Roy (1961), Szebehely and Giacaglia (1964) and Delva (1983). However, for positive eccentricities $e > 0$, the boundaries of space of motion have only been found numerically by Dvorak (1984, 1986) for Planet-type orbits surrounding both primaries (P-type orbits). In this context the limiting case $e = 1$ has been investigated separately by Kallrath (1988) both numerically and on the base of the two-fixed centre problem. For P-type orbits the lower and upper bounds for the motion of the small body in the CP are derived very accurately by Hagel (1988). This work is based on the Jacobi integral and an approximation of a second integral. Even close to the primaries the qualitative behaviour of the motion is described quite well.

In the present work we try to extend this method to the elliptic restricted three body problem [EP]. Since there exists no Jacobi integral in the EP we develop approximations of both the energy constant and the angular momentum constant.

In Sect. 2, we present the equations of motion in synodic, pulsating, barycentric coordinates. The concept of constants of motion and their differences to the integrals of motion is described in Sect. 3, where we derive two exact constants of motion for the EP. A system of two first-order integro-differential equations is transformed into two first order partial differential equations for some unknown functions V and W in Sect. 4. From that, in Sect. 5, we try to derive approximate integrals of motion for P-type orbits, i.e. those solutions of the EP which have circular initial conditions. In Sect. 6, these expressions are used to derive the lower and upper bounds for the motion of the small body. Our numerical integrator is checked against the numerical data calculated by Dvorak (1984, 1986). We also discuss the dependence of our bounds on the mass ratio and the initial angle φ_0.

2. Equations of motions

The equations of motion for the infinitesimal third body in the EP are set up in a synodic, pulsating, barycentric coordinate system (ξ, η) (Szebehely and Giacaglia 1964; Delva, 1983):

$$\xi'' - 2\eta' = \frac{\partial \omega}{\partial \xi}, \qquad \eta'' + 2\xi' = \frac{\partial \omega}{\partial \eta} \quad ' := \frac{d}{df} \tag{2.1}$$

$$\omega(\xi, \eta, f) := \frac{\Omega(\xi, \eta)}{1 + e \cdot \cos f}, \tag{2.2}$$

$$\Omega(\xi, \eta) = \frac{1}{2} \cdot [\xi^2 + \eta^2 + \mu \cdot (1-\mu)] + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \tag{2.3}$$

$$\rho_1 = [(\xi - \mu)^2 + \eta^2]^{1/2}, \qquad \rho_2 = [(\xi - \mu + 1)^2 + \eta^2]^{1/2} \tag{2.4}$$

Evaluating the derivatives the equations of motion may be written as:

$$\xi'' - 2\eta' = \frac{1}{1 + e \cdot \cos f} \cdot \left[\xi \cdot \left(1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3} \right) + \mu \cdot (1-\mu) \cdot [\rho_1^{-3} - \rho_2^{-3}] \right]$$

$$\eta'' + 2\xi' = \frac{1}{1 + e \cdot \cos f} \cdot \left[\eta \cdot \left(1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3} \right) \right]. \tag{2.6}$$

3. Derivation of two exact constants of motion

First, we would like to point out that we distinguish between integrals of motion [IOM] and constants of motion [COM]. We use the term IOM in the sense of Landau and Lifschitz (1969). An IOM I is a function of the coordinates, momenta and time whose total derivative with respect to t vanishes for all times. The common interpretation is that I does not contain quadratures with respct to time.

In addition to the term IOM we use the expression COM. By this we define a more general relation between the dynamical

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variables which may also contain quadrature expressions which we excluded above. We still keep the requirement that its total time derivative vanishes. While an IOM reduces the order of the differential equation system a COM only transforms the differential equation to a lower order integro-differential equation.

In the following we derive such a system of integro-differential equations for the EP. By multiplying (2.1a) with ξ' and (2.2b) with η', adding, and then integrating $\int df$ in the limits $[f_0, f]$, we obtain a first constant:

$$\xi'^2 + \eta'^2 - 2\int_{f_0}^{f}\left[\xi'\cdot\frac{\partial\omega}{\partial\xi} + \eta'\cdot\frac{\partial\omega}{\partial\eta}\right]\cdot df = C_1. \tag{3.1}$$

Since

$$\int_{f_0}^{f}\left[\xi'\cdot\frac{\partial\omega}{\partial\xi} + \eta'\cdot\frac{\partial\omega}{\partial\eta}\right]\cdot df = \int_{f_0}^{f}\left[\frac{d\omega}{df} - \frac{\partial\omega}{\partial f}\right]\cdot df \tag{3.2}$$

the result is that derived by Jacobi (Delva, 1983)

$$\xi'^2 + \eta'^2 - 2\cdot\omega(\xi,\eta,f) + 2\cdot\int_{f_0}^{f}\frac{\partial\omega}{\partial f}\cdot df = C_1. \tag{3.3}$$

Similarly, we multiply (2.1a) with η and (2.2b) with ξ, and subtract and integrate again $\int df$ in the limits $[f_0, f]$

$$\xi'\eta - \eta'\xi - \eta^2 - \xi^2 - \mu\cdot(1-\mu) \times \int_{f_0}^{f}\frac{1}{1+e\cdot\cos f}\cdot\eta\cdot[\rho_1^{-3} - \rho_2^{-3}]\cdot df = C_2. \tag{3.4}$$

The system (3.3, 3.4) of first order integro-differential equations is our base for further analysis. First, let us transform (ξ, η) to pulsating, polar coordinates (r, φ):

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} = r\cdot\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix},$$

$$\begin{bmatrix}\xi'\\ \eta'\end{bmatrix} = \begin{bmatrix}\cos\varphi & -r\cdot\sin\varphi\\ \sin\varphi & +r\cdot\cos\varphi\end{bmatrix}\cdot\begin{bmatrix}r'\\ \varphi'\end{bmatrix}, \tag{3.5}$$

$$\xi'\eta - \eta'\xi = -r^2\cdot\varphi',$$

$$\begin{bmatrix}r'\\ \varphi'\end{bmatrix} = \frac{1}{r}\cdot\begin{bmatrix}r\cdot\cos\varphi & +r\cdot\sin\varphi\\ -\sin\varphi & +\cos\varphi\end{bmatrix}\cdot\begin{bmatrix}\xi'\\ \eta'\end{bmatrix} \tag{3.6}$$

$$\xi'^2 + \eta'^2 = r'^2 + r^2\cdot\varphi'^2 \tag{3.7}$$

$$r'^2 + r^2\cdot\varphi'^2 - 2\omega(r,\varphi,f) + 2\cdot\int_{f_0}^{f}\frac{\partial\omega(r,\varphi,f)}{\partial f}\cdot df = \alpha \tag{3.8}$$

$$r^2 + r^2\cdot\varphi' + \mu\cdot(1-\mu)\cdot\int_{f_0}^{f}\frac{r\cdot\sin\varphi}{1+e\cdot\cos f}\cdot[\rho_1^{-3} - \rho_2^{-3}]\cdot df = \beta \tag{3.9}$$

where α and β are derived from the initial conditions $[r_0, \varphi_0, r'_0, \varphi'_0]$:

$$\alpha := r_0'^2 + r_0^2\cdot\varphi_0'^2 - 2\cdot\omega(r_0,\varphi_0,f_0),\quad \beta := r_0^2\cdot\varphi'_0 + r_0^2 \tag{3.10}$$

For further investigations it is necessary to know $[\xi, \eta, \xi', \eta']$ $(f = f_0)$. In inertial coordinates, and in the case of P-type orbits, the initial conditions at time $t = 0$ are:

$$[x_0, y_0, \dot{x}_0, \dot{y}_0] = a\cdot[\cos\varphi_0, \sin\varphi_0, -n\cdot\sin\varphi_0, n\cos\varphi_0],$$
$$n^2 = a^{-3} \tag{3.11}$$

The transformation to rotating and pulsating coordinates (ξ, η) is described in the appendix and leads to

$$[\xi_0, \eta_0] = \frac{1}{1-e}\cdot[x_0, y_0], \tag{3.12}$$

$$[\xi'_0, \eta'_0] = a\cdot(\eta - \delta)\cdot[-\sin\varphi_0, \cos\varphi_0],$$
$$\delta := \frac{(1+e\cdot\cos f_0)^2}{(1-e^2)^{1.5}}. \tag{3.13}$$

4. Relations between COM and IOM

If we succeeded to transform the quadrature expressions in the two COM's (3.8), (3.9), to functions of the same variables but not containing quadratures any more we had found two integrals of motion. Hence we require

$$\int_{f_0}^{f}\frac{\partial\omega}{\partial f}\cdot df \overset{!}{=} V(r,\varphi,f) - V(r_0,\varphi_0,f_0) = V - V_s \tag{4.1}$$

$$\int_{f_0}^{f}\frac{r\cdot\sin\varphi}{1+e\cdot\cos f}[\rho_1^{-3} - \rho_2^{-3}]\cdot df \overset{!}{=} W(r,\varphi,f) - W(r_0,\varphi_0,f_0) = W - W_s \tag{4.2}$$

We are quite aware of the fact that the existence of the functions V and W is not guarantied for any set of the initial conditions represented by α and β (3.10). After deriving equations for V and W we will discuss the existence problem in more details.

The total derivatives d/df on both sides of the above equations are lead to

$$\frac{\partial\omega}{\partial f} = \frac{\partial V}{\partial r}\cdot r' + \frac{\partial V}{\partial\varphi}\cdot\varphi' + \frac{\partial V}{\partial f},\quad \frac{\partial\omega}{\partial f} = e\cdot\sin f\cdot\frac{\Omega(r,\varphi)}{(1+e\cdot\cos f)^2} \tag{4.3}$$

$$(1+e\cdot\cos f)^{-1}\cdot r\cdot\sin\varphi\cdot[\rho_1^{-3} - \rho_2^{-3}] = \frac{\partial W}{\partial r}\cdot r' + \frac{\partial W}{\partial\varphi}\cdot\varphi' + \frac{\partial W}{\partial f} \tag{4.4}$$

$$\Omega(r,\varphi) = \frac{1}{2}\cdot[r^2 + \mu\cdot(1-\mu)] + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \tag{4.5}$$

$$\Omega(r,\varphi) = \frac{1}{2}\cdot[r^2 + \mu\cdot(1-\mu)] + \frac{1-\mu}{r}\cdot[1 - 2\lambda\cdot\cos\varphi + \lambda^2]^{-1/2} + \frac{\mu}{r}\cdot[1 + 2\nu\cdot\cos\varphi + \nu^2]^{-1/2} \tag{4.6}$$

where

$$\lambda := \mu/r,\qquad \nu := (1-\mu)/r. \tag{4.7}$$

Thus

$$\frac{dV}{df} = e\cdot\frac{\sin f}{(1+e\cdot\cos f)^2}\cdot\left[\frac{r^2 + \mu\cdot(1-\mu)}{2} + \frac{1-\mu}{r}\cdot[1 - 2\lambda\cdot\cos\varphi + \lambda^2]^{-1/2} + \frac{\mu}{r}\cdot[1 + 2\nu\cdot\cos\varphi + \nu^2]^{-1/2}\right] \tag{4.8}$$

$$\frac{dW}{df} = r^{-2} \cdot \frac{\sin\varphi}{1 + e\cdot\cos f} \cdot [(1 - 2\lambda\cdot\cos\varphi + \lambda^2)^{-3/2} - (1 + 2\nu\cdot\cos\varphi + \nu^2)^{-3/2}]. \quad (4.9)$$

Requiring that V and W are both functions of the variables r, φ and f we get two first order partial differential equations for V and W

$$r' \cdot \frac{\partial V}{\partial r} + \frac{\partial V}{\partial \varphi}\cdot\varphi' + \frac{\partial V}{\partial f} = e \cdot \frac{\sin f}{(1 + e\cdot\cos f)^2} \cdot \Omega(r, \varphi), \quad (4.10)$$

$$r' \cdot \frac{\partial W}{\partial r} + \frac{\partial W}{\partial \varphi}\cdot\varphi' + \frac{\partial W}{\partial f} = r \cdot \frac{\sin\varphi}{1 + e\cdot\cos f} \cdot [\rho_1^{-3} - \rho_2^{-3}]. \quad (4.11)$$

If for a given set of initial conditions $[r_0, \varphi_0, r'_0, \varphi'_0]$ there exist solutions for V and W we succeeded in generating IOM's of COM's. If there exists a solution then the uniqueness follows from the ansatz (4.1, 4.2) since after (4.1, 4.2) there occur only equivalent transformations. We are not able to make any global statements about the existence of V and W. However, in the next chapter we try to discuss this problem locally and for a specific type of solutions for the EP.

5. Derivation of approximate integrals for P-type orbits

We now consider the special case of direct circular solutions known as P-type orbits. Therefore the initial conditions (3.11) would lead to exact circular solutions in inertial coordinates if the two primaries were both located in the barycentre of the system or if the initial distance a tends towards infinity. We now try to solve [(4.10), (4.11)] approximately for this type of solutions.

The basic idea is to find asymptotic expressions for r' and φ' as functions of the argument r, φ, f in the case of $a \to \infty$. This would transform [(4.10), (4.11)] into a decoupled linear system of two first-order partial differential equations. From the relations between inertial coordinates (x, y) and pulsating coordinates (see appendix):

$$(\xi, \eta) = \frac{1 + e\cdot\cos f}{1 + e^2} \cdot (x, y) \quad (5.1)$$

we find the relation between $r_I = (x^2 + y^2)^{1/2}$ and $r = (\xi^2 + \eta^2)^{1/2}$ as

$$r = \frac{1 + e\cdot\cos f}{1 - e^2} \cdot r_I, \qquad r_I(f=0) = (1 - e)\cdot r(f=0). \quad (5.2)$$

Differentiating (5.2) with respect to f we obtain

$$\frac{dr}{df} = r' = -\frac{e\cdot\sin f}{1 - e^2}\cdot r_I + \frac{1 + e\cdot\cos f}{1 - e^2}\cdot r'_I. \quad (5.3)$$

Since r_I becomes the circular solution in the asymptotic case, r'_I vanishes and together with (5.2) the asymptotic expression for r' is

$$\lim_{a\to\infty} r' = -\frac{e\cdot r\cdot\sin f}{1 - e\cdot\cos f}. \quad (5.4)$$

Inspecting the integral Eq. (3.9) together with the identities (2.4) we realize that the integrand occurring in (3.9) is of the order r^{-2}, i.e. it vanishes asymptotically as $r \to \infty$. Thus, for large r, φ' becomes:

$$\lim_{r\to\infty} \varphi' = (\beta - r^2)\cdot r^{-2}. \quad (5.5)$$

Finally, we have to clarify if $a \to \infty$ induces $r \to \infty$. In other words, we must ensure that in pulsating coordinates r remains large if a is large. This can be done easily by solving the asymptotic equation (5.4) for $r(f)$:

$$\ln\frac{r}{C} = \int -\frac{e\cdot\sin f}{1 + e\cdot\cos f}\cdot df \Rightarrow r(f) = a\cdot\frac{1 + e\cdot\cos f}{1 - e} \quad (5.6)$$

We see that r is enclosed between the limiting values $r_I = a$ and $r_2 = a\cdot(1 + e/1 - e)$ so that indeed $a \to \infty$ induces $r \to \infty$. Thus the asymptotic equations for $V(r, \varphi, f)$ and $W(r, \varphi, f)$ are

$$-\frac{e\cdot r\cdot\sin f}{1 + e\cdot\cos f}\cdot\frac{\partial V}{\partial r} + \frac{\beta - r^2}{r^2}\cdot\frac{\partial V}{\partial\varphi} + \frac{\partial V}{\partial f} = e\cdot\frac{\sin f}{(1\cdot e + \cos f)^2}\cdot\Omega(r, \varphi) \quad (5.7)$$

$$-\frac{e\cdot r\cdot\sin f}{1 + e\cdot\cos f}\cdot\frac{\partial W}{\partial r} + \frac{\beta - r^2}{r^2}\cdot\frac{\partial W}{\partial\varphi} + \frac{\partial W}{\partial f} = r\cdot\frac{\sin\varphi}{1 + e\cdot\cos f}\cdot[\rho_1^{-3} - \rho_2^{-3}]. \quad (5.8)$$

Solving these equations and inserting $V(r, \varphi, f)$ and $W(r, \varphi, f)$ into (3.8) and (3.9) leads to two asymptotic integrals.

5.1. *Perturbative solution of the partial differential equations*

In principle it is possible to solve the homogeneous parts of (5.7, 5.8) exactly using the characteristics-method. The inhomogeneous equation then may be solved by applying a method of variation of constants. But unfortunately the system of first order differential equation for the characteristics

$$dr : d\varphi : df = -\frac{e\cdot r\cdot\sin f}{1 + e\cdot\cos f} : (\beta - r^2)\cdot r^{-2} : 1, \quad (5.9)$$

although it can be reduced to quadratures, leads to very complex expression and permits no closed-form solution for the inhomogeneous equations. So we decided to use a perturbative approach by expanding $V(r, \varphi, f)$ and $W(r, \varphi, f)$ in power series w.r.t. the eccentricity e:

$$V = \sum_{n=0}^{\infty} V_n\cdot e^n, \qquad W = \sum_{n=0}^{\infty} W_n\cdot e^n. \quad (5.10)$$

Inserting (5.10) into (5.7, 5.8), expanding the coefficient of $\partial V/\partial r$ as well as the forcing-terms with respect to e and comparing like powers in e results in a recursive system of equations for the contributions $V_n(r, \varphi, f)$ and $W_n(r, \varphi, f)$

$$V_0 = 0 \quad (5.11)$$

$$\frac{\beta - r^2}{r^2}\cdot\frac{\partial V_1}{\partial\varphi} + \frac{\partial V_1}{\partial f} = \sin f\cdot\Omega(r, \varphi) \quad (5.12)$$

$$\frac{\beta - r^2}{r^2}\cdot\frac{\partial V_2}{\partial\varphi} + \frac{\partial V_2}{\partial f} = -2\cdot\sin f\cdot\cos f\cdot\Omega(r, \varphi) + r\cdot\sin f\cdot\frac{\partial V_1}{\partial r} \quad (5.13)$$

$$\frac{\beta-r^2}{r^2}\cdot\frac{\partial V_3}{\partial\varphi}+\frac{\partial V_3}{\partial f}=+3\cdot\sin f\cdot\cos^2 f\cdot\Omega(r,\varphi)+r\cdot\sin f\cdot\frac{\partial V_2}{\partial r}-r\cdot\sin f\cdot\cos f\cdot\frac{\partial V_1}{\partial r}\tag{5.14}$$

$$\frac{\beta-r^2}{r^2}\cdot\frac{\partial V_N}{\partial\varphi}+\frac{\partial V_N}{\partial f}=(-1)^N\cdot N\cdot\sin f\cdot\cos^{N-1}f\cdot\Omega(r,\varphi)+r\cdot\sin f\cdot\times\sum_{k=0}^{\infty}(-1)^{N-k-1}\cdot\cos^{N-k-1}f\cdot\frac{\partial V_k}{\partial r},\quad N>0\tag{5.15}$$

$$\frac{\beta-r^2}{r^2}\cdot\frac{\partial W_0}{\partial\varphi}+\frac{\partial W_0}{\partial f}=r\cdot\sin\varphi\cdot[\rho_1^{-3}-\rho_2^{-3}]\tag{5.16}$$

$$\frac{\beta-r^2}{r^2}\cdot\frac{\partial W_1}{\partial\varphi}+\frac{\partial W_1}{\partial f}=-r\cdot\sin\varphi\cdot\cos f\cdot[\rho_1^{-3}-\rho_2^{-3}]+r\cdot\sin f\cdot\frac{\partial W_0}{\partial r}\tag{5.17}$$

$$\frac{\beta-r^2}{r^2}\cdot\frac{\partial W_2}{\partial\varphi}+\frac{\partial W_2}{\partial f}=2\cdot r\cdot\sin\varphi\cdot\cos^2 f\cdot[\rho_1^{-3}-\rho_2^{-3}]+r\cdot\sin f\cdot\frac{\partial W_1}{\partial r}-r\cdot\sin f\cdot\cos f\cdot\frac{\cdot W_0}{\partial r}\tag{5.18}$$

$$\frac{\beta-r^2}{r^2}\cdot\frac{\partial W_N}{\partial\varphi}+\frac{\partial W_N}{\partial f}=(-1)^N\cdot N\cdot r\cdot\sin\varphi\cdot\cos^{N-1}f\cdot[\rho_1^{-3}-\rho_2^{-3}]+r\cdot\sin\cdot f\sum_{k=0}^{\infty}(-1)^{N-k-1}\times\cos^{N-k-1}f\cdot\frac{\partial W_k}{\partial r},\quad N>0.\tag{5.19}$$

5.1.1. Perturbative evaluation of $W(r,\varphi,f)$

Since the forcing-term of Eq. (5.16) for W_0 does not contain f explicitly, (5.16) reduces to an ordinary differential equation:

$$\frac{\beta-r^2}{r^2}\cdot\frac{dW_0}{d\varphi}=r\cdot\sin\varphi\cdot[\rho_1^{-3}-\rho_2^{-3}]\tag{5.20}$$

To bring the solution in a simple form we make use of the fact that the right hand side of (5.20) is periodic in φ with period 2π and thus can be expressed by a Fourier-series:

$$\frac{\beta-r^2}{r^2}\cdot\frac{dW_0}{d\varphi}=\sum_{m=1}^{\infty}b_m\cdot\sin m\varphi,\qquad b_m=\frac{r}{\pi}\cdot\int_0^{2\pi}\sin m\varphi\cdot\sin\varphi\cdot[\rho_1^{-3}-\rho_2^{-3}]\,d\varphi\tag{5.21}$$

Then W_0 is

$$W_0(r,\varphi)=-\frac{r^2}{\beta-r^2}\cdot\sum_{m=1}^{\infty}\frac{1}{m}\cdot b_m\cdot\cos m\varphi.\tag{5.22}$$

Alternatively, we may give a closed-form expression for W_0 containing no more infinite sum. Thus we rewrite the term $[\rho_1^{-3}-\rho_2^{-3}]$ using (4.7, 4.9):

$$[\rho_1^{-3}-\rho_2^{-3}]=r^{-3}\cdot[(1-2\lambda\cos\varphi+\lambda^2)^{-3/2}+(1+2\nu\cdot\cos\varphi+\nu^2)^{-3/2}].\tag{5.23}$$

Then we integrate (5.20) once with respect to φ to be lead to:

$$W_0(r,\varphi)=-\frac{1}{\beta-r^2}\cdot\left[\frac{1}{\nu}\cdot(1-2\lambda\cdot\cos\varphi+\lambda^2)^{-1/2}+\frac{1}{\lambda}\cdot(1+2\nu\cdot\cos\varphi+\nu^2)^{-1/2}\right].\tag{5.24}$$

Since in later application only $W(r,\varphi,f)-W(r_0,\varphi_0,f_0)$ is used we need not care about the additive integration constant generated by integrating (5.20). Anyway, for evaluating higher order contributions in W, the Fourier-representation of W_0 (5.22) is much better adapted to our problem.

Now (5.22) is inserted into the forcing-term of (5.17). After evaluation of $\partial W_0/\partial r$, making use of some trigonometric identities and finally rearranging all terms (5.17) becomes

$$\frac{\beta-r^2}{r^2}\cdot\frac{\partial W_1}{\partial\varphi}+\frac{\partial W_1}{\partial f}=A_m\cdot\sin(m\varphi+f)+B_m\cdot\sin(m\varphi-f)\tag{5.25}$$

with

$$A_m=-\frac{1}{2}\cdot b_m-\frac{r^2}{2m}\cdot\left[\frac{2\cdot(\beta-r^2)+2r^2}{(\beta-r^2)^2}\cdot b_m+\frac{r}{\beta-r^2}\cdot\frac{db_m}{dr}\right]\tag{5.26}$$

$$B_m=-\frac{1}{2}\cdot b_m+\frac{r^2}{2m}\cdot\left[\frac{2\cdot(\beta-r^2)+2r^2}{(\beta-r^2)^2}\cdot b_m+\frac{r}{\beta-r^2}\cdot\frac{db_m}{dr}\right].\tag{5.27}$$

Since the coefficients of $\partial W_1/\partial\varphi$ and $\partial W_1/\partial f$ do not contain φ and f explicitly we may use a direct Fourier-ansatz for W_1:

$$W_1=a_m\cdot\cos(m\varphi+f)+\beta_m\cdot\cos(m\varphi-f)\tag{5.28}$$

Inserting (5.28) into (5.25), evaluating the derivatives and comparing for $\sin(m\varphi+f)$ and $\sin(m\varphi-f)$ finally results in the Fourier-representation of W_1:

$$W_1(r,\varphi,f)=-r^2\cdot\sum_{m=1}^{\infty}\frac{A_m\cdot\cos(m\varphi+f)}{r^2+(\beta-r^2)\cdot m}-\frac{B_m\cdot\cos(m\varphi-f)}{r^2-(\beta-r^2)\cdot m}.\tag{5.29}$$

It turned out to be sufficient for the accuracy of the approximate integrals of motion to terminate the evaluation of W with W_1. To be complete we add an approximate analytic evaluation of the Fourier-components b_m. The method is to expand (5.23) into a Taylor-series with respect to λ and ν to a given order. This automatically leads to a trigonometric series in $\sin m\varphi$ from which the coefficients b_m to the specified order in λ and ν can be directly extracted (see Hagel, 1988). To the 4-th order in λ and ν the coefficients are:

$$b_1=r^{-2}\cdot[\tfrac{3}{8}\cdot(\lambda^2-\nu^2)+\tfrac{15}{64}\cdot(\lambda^4-\nu^4)]\tag{5.30}$$

$$b_2=r^{-2}\cdot[\tfrac{3}{8}\cdot(\lambda^2-\nu^2)+\tfrac{15}{64}\cdot(\lambda^3-\nu^3)]\tag{5.31}$$

$$b_3=r^{-2}\cdot[\tfrac{15}{8}\cdot(\lambda^2-\nu^2)+\tfrac{105}{128}\cdot(\lambda^4+\nu^4)]\tag{5.32}$$

$$b_4=r^{-2}\cdot[\tfrac{35}{16}\cdot(\lambda^3+\nu^3)]\tag{5.33}$$

$$b_5=r^{-2}\cdot[\tfrac{315}{128}\cdot(\lambda^4-\nu^4)]\tag{5.34}$$

The factors db_m/dr can be obtained by differentiating (5.30)–(5.34) with respect to r keeping in mind that $\lambda = \mu/r$ and $\nu = (1-\mu)/r$.

5.1.2. Perturbative evaluation of $V(r,\varphi,f)$

We consider first the Eq. (5.12) for $V_1(r,\varphi,f)$. As above using the periodicity of $\Omega(r,\varphi)$ in φ we may rewrite (5.12) in terms of a Fourier-series

$$\Omega(r,\varphi) = \sum_{m=0}^{\infty} d_m \cdot \cos m\varphi. \tag{5.35}$$

Then with

$$\sin f \cdot \cos m\varphi = \tfrac{1}{2}\cdot[\sin(m\varphi + f) - \sin(m\varphi - f)] \tag{5.36}$$

we obtain

$$\frac{\beta - r^2}{r^2}\cdot\frac{\partial V_1}{\partial\varphi} + \frac{\partial V_1}{\partial f} = \frac{1}{2}\cdot\sum_{m=0}^{\infty} d_m[\sin(m\varphi+f) + \sin(m\varphi - f)]. \tag{5.37}$$

With the ansatz

$$V_1 = g_m\cdot\cos(m\varphi+f) + h_m\cdot\cos(m\varphi - f) \tag{5.38}$$

we are lead to

$$V_1 = -\frac{r^2}{2}\cdot\sum_{m=0}^{\infty}\left[\frac{\cos(m\varphi+f)}{r^2+(\beta-r^2)\cdot m} - \frac{\cos(m\varphi-f)}{r^2-(\beta-r^2)\cdot m}\right]\cdot d_m. \tag{5.39}$$

The contribution V_2 (Eq. (5.12)) is split into two parts $V_2 = V_{21} + \bar{V}$ with

$$\frac{\beta - r^2}{r^2}\cdot\frac{\partial V_{21}}{\partial\varphi} + \frac{\partial V_{21}}{\partial f} = -\sin 2f\cdot\Omega(r,\varphi) \tag{5.40}$$

$$\frac{\beta - r^2}{r^2}\cdot\frac{\partial \bar{V}}{\partial\varphi} + \frac{\partial \bar{V}}{\partial f} = r\cdot\sin f\,\frac{\partial V_1}{\partial r}. \tag{5.41}$$

Comparing (5.40) and (5.12) we find

$$V_{21} = -\frac{r^2}{2}\cdot\sum_{m=0}^{\infty}\left[\frac{\cos(m\varphi+2f)}{2r^2+(\beta-r^2)\cdot m} - \frac{\cos(m\varphi-2f)}{2r^2-(\beta-r^2)\cdot m}\right]\cdot d_m \tag{5.42}$$

For $\bar{V}$ we derive:

$$\bar{V} = -\frac{A_0 - B_0}{2} - r^2\cdot\sum_{m=1}^{\infty}\left[\frac{A_m\cdot\cos(m\varphi+2f)}{2r^2+(\beta-r^2)\cdot m} - \frac{B_m\cdot\cos(m\varphi-2f)}{2r^2-(\beta-r^2)m} + \frac{C_m\cdot\cos m\varphi}{(\beta-r^2)\cdot m}\right] \tag{5.43}$$

where

$$A_m = -\frac{r^2}{2}\cdot\frac{d_m}{r^2+(\beta-r^2)\cdot m} - \frac{r^3}{4}\cdot\frac{d'_m}{r^2+(\beta-r^2)\cdot m} + \frac{r^4}{2}\cdot\frac{d_m(1-m)}{[r^2-(\beta-r^2)\cdot m]^2} \tag{5.44}$$

$$B_m = +\frac{r^2}{2}\cdot\frac{d_m}{r^2+(\beta-r^2)\cdot m} + \frac{r^3}{4}\cdot\frac{d'_m}{r^2+(\beta-r^2)\cdot m} - \frac{r^4}{2}\cdot\frac{d_m\cdot(1-m)}{[r^2-(\beta-r^2)\cdot m]^2} \tag{5.45}$$

$$C_m = -\frac{r^4}{r^4+(\beta-r^2)^2\cdot m^2}\cdot\left[d_m + \frac{r^2}{2}\cdot d'_m\right] + \frac{r^4}{2}d_m \times\left[\frac{1-m}{[r^2-(\beta-r^2)\cdot m]^2} + \frac{1+m}{[r^2-(\beta-r^2)\cdot m]^2}\right]. \tag{5.46}$$

The Fourier-coefficients d_m are found as before by expanding $\Omega(r,\varphi)$ into a Taylor-series with respect to $\lambda = \mu/r$ and $\nu = (1-\mu)/r$. To 4-th order in λ and ν we get:

$$d_0 = \tfrac{1}{2}\cdot[r^2 + \mu\cdot(1-\mu)] + \nu\cdot[1 + \tfrac{1}{4}\cdot\lambda^2 + \tfrac{9}{64}\cdot\lambda^4] + \lambda\cdot[1 + \tfrac{1}{4}\cdot\lambda^2 + \tfrac{9}{64}\cdot\lambda^4] \tag{5.47}$$

$$d_1 = \tfrac{3}{8}\cdot\lambda\cdot\nu\cdot(\lambda - \nu) \tag{5.48}$$

$$d_2 = \nu\cdot\lambda^2\cdot[\tfrac{3}{4} + \tfrac{5}{16}\cdot\lambda^2] + \lambda\cdot\nu^2\cdot[\tfrac{3}{4} + \tfrac{5}{16}\cdot\lambda^2] \tag{5.49}$$

$$d_3 = \tfrac{5}{8}\cdot[\nu\cdot\lambda^3 - \lambda\cdot\nu^3] \tag{5.50}$$

$$d_4 = \tfrac{35}{64}\cdot[\nu\cdot\lambda^4 - \lambda\cdot\nu^4] \tag{5.51}$$

The d'_m are again found by differentiating the d_m with respct to r.

5.1.2.1. Simplification of V and perturbation results up to 4-th order in e

A considerable simplification in the perturbation expansion of $V(r,\varphi,f)$ is possible if we inspect the Fourier-coefficients d_m of $\Omega(r,\varphi)$ [Eqs. (5.47)–(5.51)]. While the leading power of r in d_0 is r^2 it is only r^{-2} in d_1, r^{-3} in d_2 a.s.o. So there is at least a difference of 4 in the order of r between d_0 and $d_{m>0}$ which means that in the asymptotic case $r\to\infty$ which we deal with d_0 clearly dominates all $d_{m>0}$. As a consequence of this fact we may drastically simplify the cumbersome expressions for V_1 and V_2 obtained in the previous section by neglecting all $d_{m>0}$. It even turns out that the accuracy of our results is not affected if we take only into account terms down to order r^{-1} in d_0. With all these simplifications we finally arrive at:

$$d_0 = \tfrac{1}{2}\cdot[r^2 + \mu\cdot(1-\mu)] + r^{-1} + O(r^{-3}),\quad d'_0 = r + O(r^{-2}) \tag{5.52}$$

By making $d_{m>0} = 0$ in Eqs. (5.39), (5.42) and (5.43) V_1 and V_2 are reduced to

$$V_1 = -d_0\cdot\cos f \tag{5.53}$$

$$V_2 = \frac{1}{2}\cdot\left[d_0 + \frac{r^2}{2}\right]\cdot\cos 2f. \tag{5.54}$$

By inspecting Eqs. (5.14, 5.15) for $N = 4$ we realize that with the described simplifications even V_3 and V_4 can be written in a compact form:

$$V_3 = -\tfrac{1}{4}\cdot[3\cdot d_0 - r^2]\cdot\cos f - \tfrac{1}{4}\cdot[d_0 + r^2]\cdot\cos 3f \tag{5.55}$$

$$V_4 = \tfrac{1}{2}\cdot d_0\cdot\cos 2f + [\tfrac{1}{8}\cdot d_0 + \tfrac{3}{16}\cdot r^2]\cdot\cos 4f. \tag{5.56}$$

For $W(r,\varphi,f)$ these simplifications are not possible since the difference in orders of r between b_1 and b_2 is only 1 instead of 4 in case of the d_m.

5.2. The two integrals of motion

Together with the expressions up to first order in e for $W(r,\varphi,f)$ [Eqs. (5.22), (5.29)] and the simplified expressions up to 4-th order in e for V [Eqs. (5.53)–(5.56)] and by using (5.10) to the

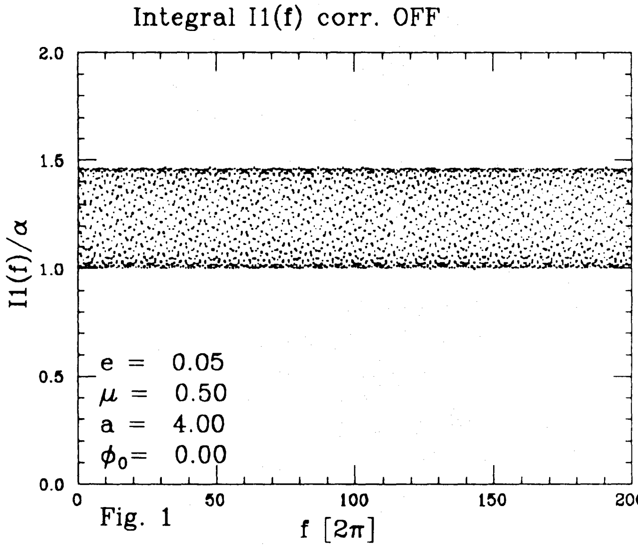

Fig. 1

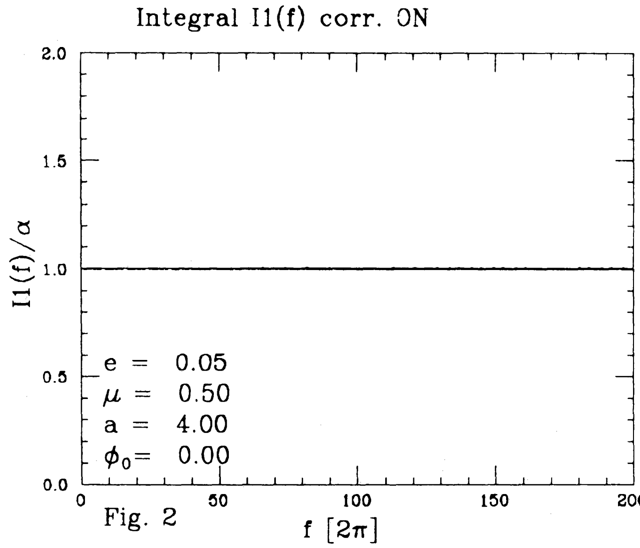

Fig. 2

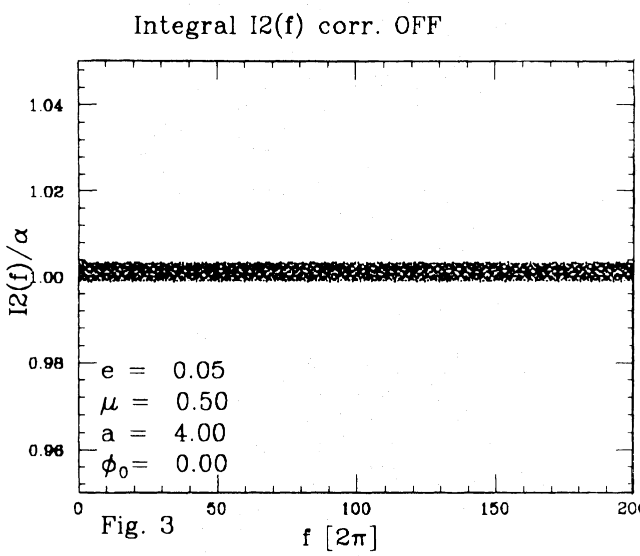

Fig. 3

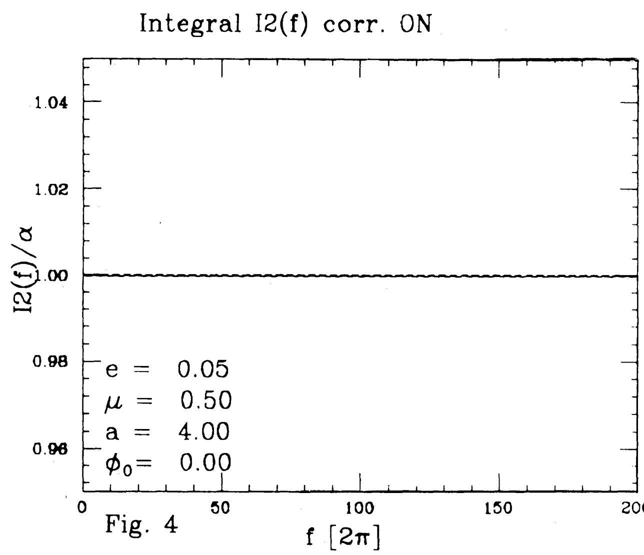

Fig. 4

Figs. 1–8. The figures show the function $I\,1/\alpha$ and $I\,2/\beta$ as functions of time. The integration was performed using the parameter set $[e, \mu, \mathrm{a}, \phi_0] = [0.05, 0.5, 4, 0]$ (upper four pictures) and $[e, \mu, a, \phi_0] = [0.2, 0.5, 4, 0]$ (lower pictures). The left columns, i.e. **Figs. 1, 3, 5, and 7**, show the results when the correction $V - V_s$, resp. $W - W_s$, was not taken into account. The amplitude of the variation of $I\,1/\alpha$ is reduced drastically due to the correction $V - V_s$ which was taken into account this time. The effect of the correction $W - W_s$ is less remarkable

given orders we finally arrive at the two approximate integrals for P-type orbits in the EP:

$$I1\colon r'^2 + r^2 \cdot \varphi'^2 - 2 \cdot \frac{\Omega(r, \varphi)}{1 + e \cdot \cos f} + 2 \cdot [V(r, \varphi, f) - V_s] = \alpha \tag{5.57}$$

$$I2\colon r^2 + r^2 \cdot \varphi' + \mu \cdot (1 - \mu) \cdot [W(r, \varphi, f) - W_s] = \beta \tag{5.58}$$

where

$$W_s := W(r_0, \varphi_0, f_0), \qquad V_s := V(r_0, \varphi_0, f_0)\,\{\text{we choose } f_0 = 0\}. \tag{5.59}$$

Of course, for $e = 0$, V vanishes ($V_0 = 0$) and we are left with the exact Jacobian integral valid in the circular restricted problem. The second integral $I2$ still is approximate for $e = 0$. It then represents exactly the generalisation of the angular momentum integral for the circular problem which in this form has been derived by Hagel (1988).

In order to check the integral properties of $I1$ and $I2$ we wrote a FORTRAN program which for a given set of the values (e, μ, a, φ_0) integrates the exact differential equations of motion (2.5, 2.6), transforms to polar coordinates $(r, \varphi, r', \varphi')$ and plugs these coordinates into (5.57, 5.58) after each integration step. The integration method used was Runge–Kutta-4 with a discretization $df = 0.05$ (i.e. more than 120 steps per revolution of the primaries). Figures 1–8 shows the results of this test for the following set of parameters:

(i) $(e, \mu, A, \varphi_0) = (0.05, 0.5, 4, 0)$

(ii) $(e, \mu, A, \varphi_0) = (0.20, 0.5, 4, 0)$

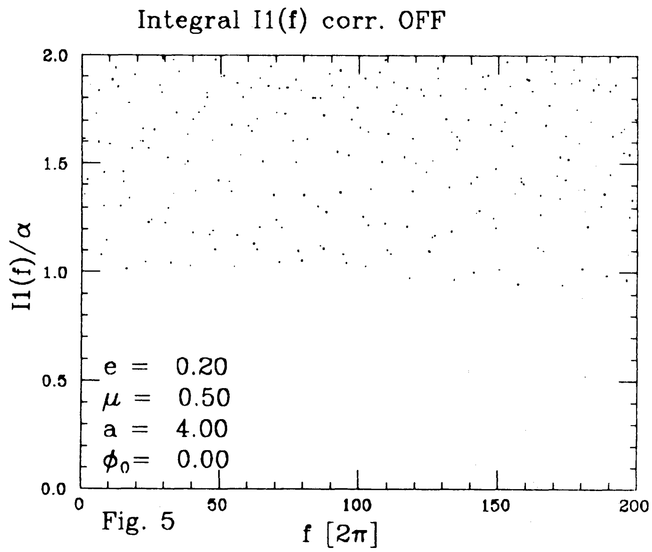

Fig. 5

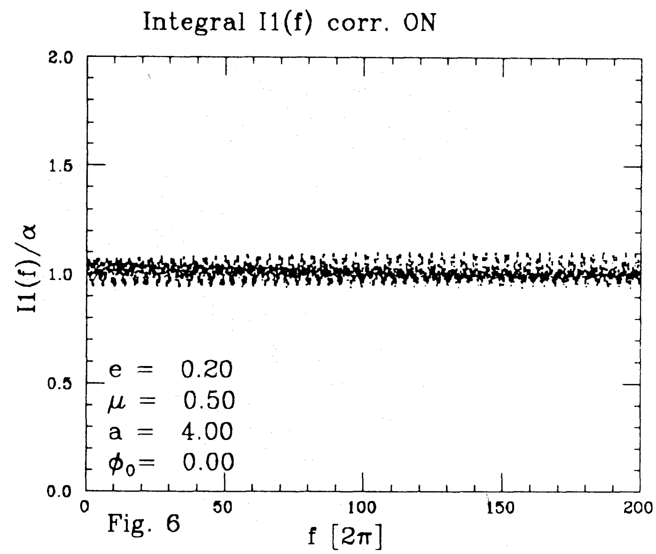

Fig. 6

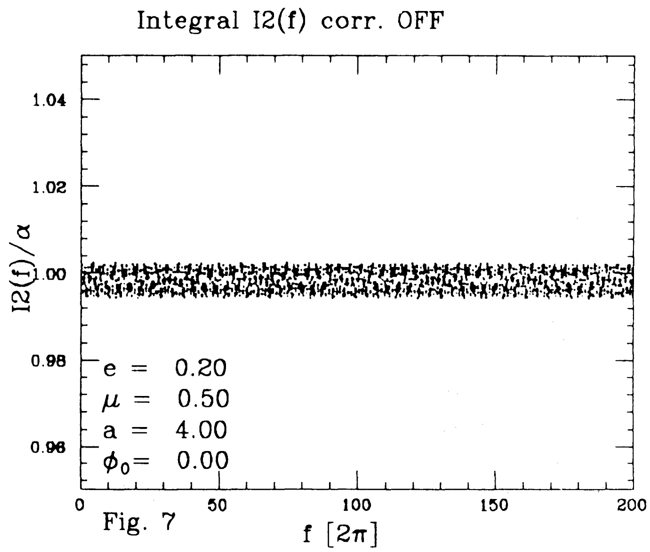

Fig. 7

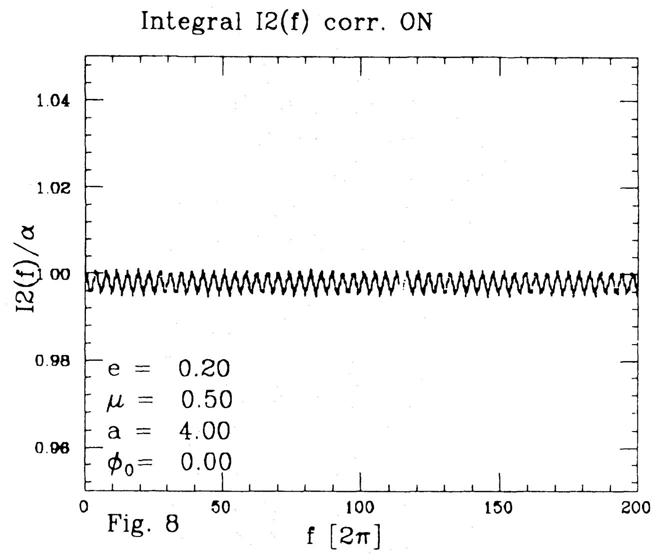

Fig. 8

Figs. 5–8

While Figs. 1–4 relate to (i) Figs. 5–8 show the results for (ii). The left figure in all cases plots the variation of $I1$ and $I2$ as a function of the number of periods of the primaries when the corrections $V - V_s$ and $W - W_s$ are not taken into account ($I1$ and $I2$ then are the pure Jacobian "Integrals" and the angular momentum resp.). The right figure shows $I1$ and $I2$ with $V - V_s$ and $W - W_s$ taken into account. In all cases we observe a clear improvement of the corrected $I1$ and $I2$ with respect to the uncorrected ones. While in the scale of $(0 < I1/\alpha, I2/\beta < 2)$ for $e = 0.05$ no variation of $I1$ and $I2$ is visible at all, for $(0 < f < 2000\pi)$ we observe a tiny residual variation of $I1/\alpha$ for the parameter set (ii). So in fact we succeeded to derive two near-integrals for P-type orbits in the EP if the initial distance a is sufficiently large ($a > 3$). Due to the perturbative approach with respect to e the valid interval of e up to now seems to be limited to $(0 < e < 0.2)$. By further expansion of V and W to higher orders in e this interval is believed to be still enlarged.

6. Region of motion

The region accessible for the third body is limited by the requirement that $\dot{r}^2$ is positive. From Eqs. (5.57, 5.58) we derive

$$r^2 \cdot \dot{r}^2 = h(r, \varphi, f) \tag{6.1}$$

where

$$h(r, \varphi, f) = r^2 \cdot [\alpha + 2 \cdot \omega(r, \varphi, f) - 2 \cdot (V - V_s)] - [\beta - \mu \cdot (1 - \mu) \cdot (W - W_s) - r^2] \tag{6.2}$$

For an analysis of the accessible region we seek the roots r_- and r_+ of $h(r, \varphi, f)$ with respect to r for given values of φ and f and the set of initial parameters $[e, \mu, a, \varphi_0]$. Let us now discuss the choice of φ and f. Although the two integrals of motion

$$I1(r, \varphi, f) = \alpha, \qquad I2(r, \varphi, f) = \beta \tag{6.3}$$

represent two functional relations between the coordinates r and φ and the independent variable f being constant, in general not all possible triplets (r, φ, f) fulfilling (6.3) really will occur after a sufficiently long time. This is due to the fact that there exist additional relations between r and f and between r and φ which are of course given by the exact (unknown) solution $r = r(f)$ and $\varphi = \varphi(f)$ being subject to the equations of motion. So for computing the regions of motion which can be covered by the solution we only use values of φ and f for which we know that they occur at the same time. In practice only one such doublet is known, namely $\varphi(f = 0) = \varphi_0$. Since P-type orbits intersect the line $\varphi = \varphi_0$ for an infinite number of times and the solution is aperiodic in general we may expect all possible r-values to occur after an infinite time so that we need not make any additional constraint for the radius r at $\varphi = \varphi_0$. So we are only able to give a statement about regions of motion for $\varphi = \varphi_0$. Inserting this into our general condition we have to investigate the distance of the zeros r_- and r_+ of $h(r, \varphi, f)$. This limits r_- and r_+ of the accessible region are investigated both numerically and analytically for a variety of parameters $[e, \mu, a, \varphi_0]$. The comparison of the bounds for different sets $[e, \mu, a, \varphi_0]$ has to be performed in the inertial frame since the values of r_- and r_+ for different parameters may correspond to different anomalies f and therefore may give a completely wrong impression. However, the derivation of analytical bounds is only possible in the pulsating coordinates. Although our analysis to find the zeros r_- and r_+ is based on $\varphi(f = 0) = \varphi_0$ we do not know the proper value of f to transform r_- and r_+ back to the inertial frame. Nevertheless we are able to give the theoretical bounds in the inertial frame by the assymption that the anomaly f is needed in the theoretical analysis is the same as that we find in the numerical integration. If this is true we transform both lower and upper bounds by

$$R^{\text{inertial}} = r^{\text{inertial}} \cdot R^{\text{pulsating}} / r^{\text{pulsating}} \tag{65}$$

where R refers to the theoretical solution and r to the numerical one.

Table 1. Dependence of the region of motion on the eccentricity e for mass ratio $\mu = 0.5$, $a_0 = 4$, $\varphi_0 = 0$ and an integration time of 200 revolutions of the primaries. The table gives the smallest and largest radius vector r_- and r_+ occurring during the integration in pulsating and inertial coordinates and the values derived from our semi-analytical approach. Both approaches lead to upper limits $R_+ = a_0$. For $e > 0.25$ the differences between the analytical and numerical results become very obvious

	Numerical integration			Semi-analytical approach		
e	r_-	r_+	R_-	r_-	r_+	R_-
0.000	3.932	4.000	3.932			
0.025	3.831	4.103	3.927	3.818	4.103	3.914
0.050	3.734	4.211	3.920	3.716	4.211	3.901
0.075	3.640	4.324	3.913	3.621	4.324	3.893
0.100	3.550	4.445	3.905	3.533	4.444	3.886
0.125	3.464	4.572	3.897	3.448	4.571	3.879
0.150	3.380	4.707	3.887	3.362	4.706	3.866
0.175	3.299	4.850	3.877	3.270	4.848	3.843
0.200	3.222	5.001	3.866	3.165	5.000	3.798
0.225	3.147	5.162	3.854	3.043	5.161	3.727
0.250	3.074	5.334	3.842	2.895	5.333	3.618
0.275	3.003	5.518	3.829	2.257	5.517	2.878
0.300	2.935	5.714	3.815	0.469	5.714	0.610

7. Conclusions

Our numerical integrator was tested by checking the constancy of the Jacobian integral in the case of vanishing eccentricity e and by checking our numerical bounds against that of Dvorak (1984, p. 371). From our numerical experiments and our semi-analytical investigation which is partly documented in Table 1–3 we list the following results:

1. The analytic results are in good agreement with numerical experiments up to eccentricities about $e = 0.25$. The analytic upper bound also fits for even higher eccentricities up to 0.5.

2. For $\mu = 0.5$ and small eccentricities, and therefore orbits which are only slightly disturbed, the upper bound in inertial coordinates coincides with the initial distance a both numerically and analytically. This situation is similar to that in the limiting case $e = 1$ investigated by Kallrath (1988).

Table 2. Dependence of the region of motion on the initial position angle φ_0 for eccentricities $e = 0.1$ and 0.2, mass ratio $\mu = 0.5$, $a_0 = 4$, and an integration time of 200 revolutions of the primaries. See Table 1 for more information. Due to symmetry, the angles $\varphi_0 + 180°$ lead to the same results as φ_0. The upper limit in pulsating coordinates r_+ is equal to $a_0/(1 - e^2)$. In the inertial frame this gives $R_+ = a_0$. The columns give our resuls for $e = 0.1$ and $e = 0.2$, respectively

	Numerical integration				Semi-analytical approach			
φ_0	$r_-(0.1)$	$r_-(0.2)$	$R_-(0.1)$	$R_-(0.2)$	$r_-(0.1)$	$r_-(0.2)$	$R_-(0.1)$	$R_-(0.2)$
0	3.550	3.222	3.905	3.866	3.533	3.165	3.886	3.798
45	3.548	3.250	3.901	3.895	3.594	3.297	3.952	3.951
90	3.543	3.269	3.897	3.923	3.652	3.407	4.017	4.089
135	3.548	3.248	3.901	3.897	3.594	3.297	3.951	3.956

Table 3. Dependence of the region of motion on the mass ratio μ for eccentricities $e = 0.1$ and 0.2, $\varphi_0 = 0$, $a_0 = 4$, and an integration time of 200 revolutions of the primaries. See Table 1 for more information. The upper limit in pulsating coordinates r_+ is equal to $a_0/(1-e^2)$. In the inertial frame this gives $R_+ = a_0$. The columns give our results for $e = 0.1$ and $e = 0.2$, respectively

	Numerical integration				Semi-analytical approach			
μ	$r_-(0.1)$	$r_-(0.2)$	$R_-(0.1)$	$R_-(0.2)$	$r_-(0.1)$	$r_-(0.2)$	$R_-(0.1)$	$R_-(0.2)$
0.1	3.564		3.916		3.601		3.977	
0.2	3.544	3.181	3.892	3.810	3.559	3.203	3.908	3.836
0.3	3.554	3.199	3.907	3.837	3.536	3.169	3.887	3.801
0.4	3.556	3.232	3.908	3.878	3.528	3.158	3.877	3.789
0.5	3.550	3.222	3.905	3.866	3.533	3.165	3.886	3.798

3. The lower bound decreases when the initial position angle φ_0 increases from 0 to 90 degrees (Table 2), i.e. the region of motion becomes broader. However, the analytical approach breaks down for $\varphi_0 = 90$ or $\varphi_0 = 270$.

4. The results do not show a strong dependence on the mass ratio μ (Table 3).

The dependence of the bounds on φ_0 and μ may become more critical for orbits with higher perturbation. The disturbance may be increased by decreasing the size of the initial distance a.

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Appendix: Transformation of coordinates and velocities

In this section let r denote the distance between the primaries. Here r is a function of the true anomaly f:

$$r = \frac{p}{1 + e\cdot\cos f}, \qquad p := a\cdot(1-e^2)$$

where A is the semi major-axis of the primaries' motion. Usually the dimensions are chosen so that A is unity.

According to Broucke (1969) the transformation of the velocities of the planet from inertial (or rotating) to pulsating coordinates is given by:

$$\xi' = y\cdot\delta + \dot{x}\cdot\cos f + \dot{y}\cdot\sin f, \qquad \delta := \delta(f, e) = \sqrt{p}\,|\cdot r^{-2}$$

$$\xi' = -x\cdot\delta - \dot{x}\cdot\sin f + \dot{y}\cdot\cos f$$

For $f = f_0$ and the initial coordinates $[x, y] = a\cdot[\cos\varphi_0, \sin\varphi_0]$ we achieve

$$\xi' = y\cdot\delta + \dot{x} = a\cdot\sin\varphi_0\,\delta - a\cdot n\cdot\sin\varphi_0, \qquad \delta := \frac{(1+e\cdot\cos f_0)^2}{(1-e^2)^{1.5}}$$

$$\xi' = -x\cdot\delta + \dot{y} = -a\cdot\cos\varphi_0\cdot\delta + a\cdot n\cdot\cos\varphi_0$$

and thus

$$[\xi'_0, \eta'_0] = a\cdot(n-\delta)\cdot[-\sin\varphi_0, \cos\varphi_0]$$

We note that in the circular problem $\delta = \delta(e=0) = 1$.

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