PACKING CONVEX POLYGONS IN MINIMUM-PERIMETER CONVEX HULLS

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Abstract. The problem of packing a given set of freely translated and rotated convex polygons in a minimum-perimeter convex polygon (in particular the minimum-perimeter convex hull) is introduced. A mathematical model of the problem using the phi-function technique is provided. Problem instances with up to 6 convex polygons are solved by the global NLP solver BARON to get a minimum-perimeter convex hull. Numerical experiments for larger instances are reported using the local NLP solver IPOPT.

Keywords: Global optimization, non-convex nonlinear programming, polygon packing problem, convex hull, perimeter minimization, non-overlap constraints, computational geometry

1 INTRODUCTION

Finding a convex hull for a given number of polygons fixed w.r.t. to position and orientation is a classical problem in computational geometry (Preparata 1985; Avis 1997; Thomas, 2001) and has important applications in packing and cutting, manufacturing, operations research, mechanics, chemistry (DeBerg, 2008; Scheithauer, 2018). The applications range from collision detecting in animation to estimating output of oil wells (DeBerg, 2008), from spatial extent of an epidemic (Dumonteil, 2013) to path finding in robotics (Scheithauer, 2018). In most applications analyses of several polygons are substituted by studying a single geometric object (convex hull). Various efficient solution techniques for this problem have been proposed (Alt, 2015; Yagiura, 2021). The problem becomes more complicated when the polygons are freely translated and rotated without mutual overlapping of their interiors. In this case different convex hulls can be constructed depending on the positioning and rotating polygons. Correspondingly, the problem to find a minimum perimeter (area) convex hull arises (Alt, 2015, Tang 2006). It can be also interpreted as packing polygonal objects in a minimal convex container (Kallrath, 2009).

Polynomial solution techniques for minimum-perimeter convex hulls are known for a few (two or three) polygons (Ahn, 2012) and/or for polygons with limited translations/rotations (Park, 2016). Numerical techniques for packing two freely translated and rotated irregular objects in a minimal polygonal container with a limited number of vertices were proposed in (Bennel 2015). Solution methods for finding a minimal perimeter convex hull of an arbitrary number of disks were considered in (Kallrath, 2019) with analytical solutions obtained for 3 disks. In (Kallrath, 2018) the minimal surface convex hull problem for spheres was studied numerically and analytically.

In this paper the problem of packing freely translated and rotated convex polygons in a minimum perimeter convex polygonal container is considered. The maximal number of vertices of the container is fixed, however the shape of the container is not specified and is defined to minimize the perimeter. More specifically, the problem is as follows: find a minimum-perimeter convex m-gon (polygon with at most m vertices) containing all given convex polygons (objects) without overlapping. Note that if m is sufficiently large (at least the total number of vertices of all polygons), the minimal perimeter m-gon provides a minimal convex hull of the polygons.

The main contributions of the paper are as follows:

- The problem of packing a given set of freely translated and rotated convex polygons in a minimum-perimeter convex m-gon is introduced.
- Analytical tools to state placement conditions for variable polygonal shape domain are presented, using the phi-function technique.
- A nonlinear optimization problem is formulated for packing convex polygons in a minimal perimeter convex polygonal container using the phi-function technique.

- A number of problem instances (with up to 6 convex polygons) are solved to optimality by the global optimizer BARON (Tawarmalani & Sahinidis, 2005) and can be used as the benchmark problems for further research.
- Numerical experiments for larger instances are reported.

The rest of the paper is organized as follows. Section 2 presents the problem formulation. Analytical tools to state placement conditions are given in Section 3, while two mathematical models, using phi-functions and quasi phi-functions, are provided in the next section. In Section 5 the main details of the mathematical models are illustrated for the case of two triangular objects. Computational results are discussed in Section 6, while Section 7 concludes. Comments on an analytical solution for the case of two triangles are presented in Appendix.

2 PROBLEM FORMULATION

Objects. Let a collection of convex polygons A_q, $q \in I_n = \{1,\dots,n\}$ be given. Each convex polygon A_q is defined by its vertices $\tilde{v}_{qi} = (\tilde{x}_{qi}, \tilde{y}_{qi})$, $i = 1,\dots,m_q$, in the local coordinate system and $A_q = \{(\tilde{x},\tilde{y}) : \tilde{\varphi}_{qi}(\tilde{x},\tilde{y}) \le 0,\ i = 1,\dots,m_q\}$, where $\tilde{\varphi}_{qi}(\tilde{x},\tilde{y}) = 0$ for $i = 1,\dots,m_q$, are equations of its sides, $\tilde{\varphi}_{qi}(x, y) = \tilde{\alpha}_{qi}x + \tilde{\beta}_{qi}y + \tilde{\gamma}_{qi}$, $\tilde{\alpha}_{qi}^2 + \tilde{\beta}_{qi}^2 = 1$ The location and orientation of A_q is defined by a variable vector of its placement parameters (x_q, y_q, θ_q). The translation of A_q by the vector $v_q = (x_q, y_q) \in R^2$ and the rotation of A_q by the angle $\theta_q \in [0, 2\pi)$ is defined by $A_q(v_q, \theta_q) =$ $= \{t \in R^2 : t = v_q + M(\theta_q)\tilde{t}, \forall \tilde{t} \in A_q(0,0,0)\}$, where $A_q(0,0,0)$ denotes the non-translated and non-rotated polygon A_q and $M(\theta_q) = \begin{pmatrix} \cos\theta_q & \sin\theta_q \\ -\sin\theta_q & \cos\theta_q \end{pmatrix}$ is the rotation matrix.

Each point $\tilde{t} = (\tilde{x}, \tilde{y}) \in A_q(0,0,0)$ in the local coordinate system of A_q is transformed into a point (x, y) given a translation of (x_q, y_q) and rotated by an angle θ_q as follows:

$$x = \tilde{x} \cdot \cos\theta_q + \tilde{y} \cdot \sin\theta_q + x_q, \quad y = -\tilde{x} \cdot \sin\theta_q + \tilde{y} \cdot \cos\theta_q + y_q .$$

Each straight line

$$\tilde{L} = \{(x, y) \in R^2 \,|\, \tilde{\alpha} x + \tilde{\beta} y + \tilde{\gamma} = 0, \tilde{\alpha}^2 + \tilde{\beta}^2 = 1\}$$

is transformed into the straight line $L = \{(x, y) \in R^2 \,|\, \alpha x + \beta y + \gamma = 0\}$,

where $\alpha = \tilde{\alpha} \cdot \cos\theta_q + \tilde{\beta} \cdot \sin\theta_q$, $\beta = -\tilde{\alpha} \cdot \sin\theta_q + \tilde{\beta} \cdot \cos\theta_q$, $\gamma = \tilde{\gamma} - \alpha \cdot x_q - \beta \cdot y_q$,

θ_q is a rotation parameter, (x_q, y_q) is a translation vector A_q.

Container. Define a collection of line segments e_k, $k = 1, \ldots, m$, considered in the fixed coordinate system OXY. Each line segment e_k is given by the variable vector

$$(\omega_k, \Theta_k, t_k), \tag{1}$$

where $\omega_k = (x_k^\omega, y_k^\omega)$ is an initial vertex, t_k is a variable length and Θ_k is a variable rotation angle of e_k with respect to OX axis (see Fig.1).

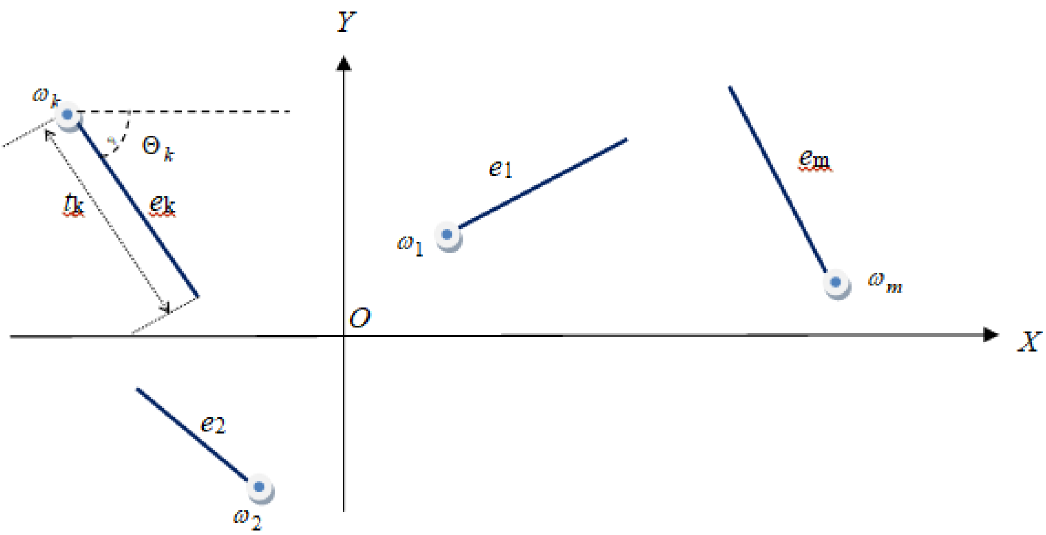


Fig. 1 A collection of line segments e_k, $k = 1, \ldots, m$

The vector of all variables of the collection of m line segments is denoted by

$\mathbf{p} = (x_k^\omega, y_k^\omega, \Theta_k, t_k, k = 1, 2, \ldots, m)$.

The objective is to find an arrangement of the line segments e_k, $k = 1, \ldots, m$, that form a minimum perimeter convex polygon having at most m vertices and containing the collection of convex polygons A_q, $q \in I_n$.

The following restrictions on the arrangement of the segments e_k, $k = 1,\dots,m$ have to be taken into account:

a cycle configuration condition

$$x^{\omega}_{k+1} = x^{\omega}_k + t_k \cdot \cos\Theta_k,\ y^{\omega}_{k+1} = y^{\omega}_k + t_k \cdot \sin\Theta_k, \tag{2}$$

for $k = 1,\dots,m$, $x^{\omega}_{m+1} = x^{\omega}_1$, $y^{\omega}_{m+1} = y^{\omega}_1$;

Further the domain bounded by the line segments e_k, $k = 1,\dots,m$ and satisfying conditions (1)-(2) is denoted by Ω (see Fig. 2).

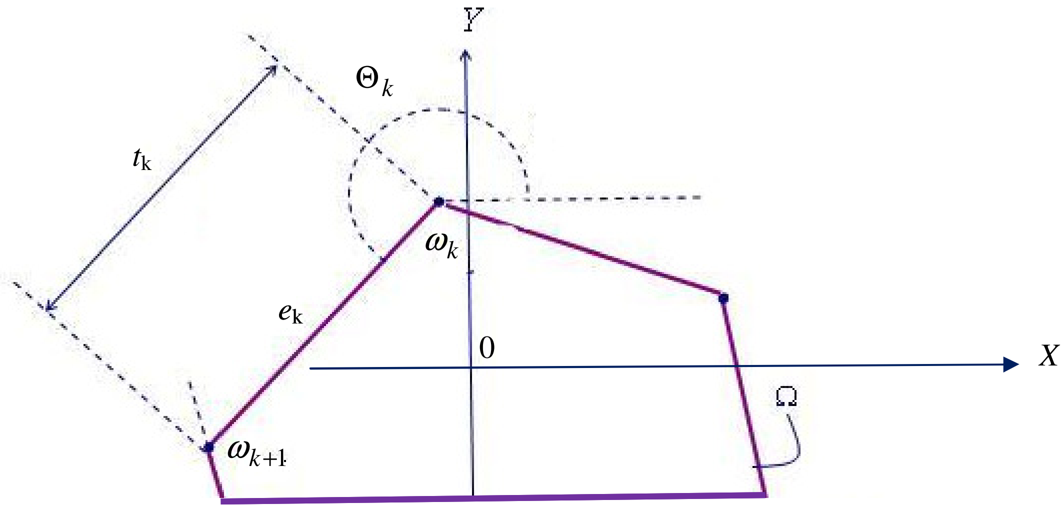


Fig. 2 Domain Ω of the variable vector $\mathbf{p} = (x^{\omega}_k, y^{\omega}_k, \Theta_k, t_k, k = 1, 2, \dots, 5)$: convex *m*-gon of variable edges e_k, $k = 1,\dots,m = 5$

A domain Ω with the variable *metrical characteristics* $\mathbf{p}$ defined above is called a variable *container* $\Omega = \Omega(\mathbf{p})$ for the collection of moving convex polygons A_q, $q \in I_n$.

The optimization problem is considered in the following setting.

Problem. Find the minimal perimeter container $\Omega(\mathbf{p}^*)$ with at most m vertices for the collection of non-overlapping convex polygons A_q, $q \in I_n$.

Remark. Note that the problem above is considered for a fixed m. The minimal perimeter container with at most m vertices in general does not provide a convex hull of the polygons. However, if m is sufficiently large (greater or iqual to the sum of m_q for all $q \in I_n$), then the minimal perimeter polygonal container is a minimal perimeter convex hull.

The following placement constraints have to be fulfilled:
non-overlapping of each pair of convex polygons $A_q(u_q)$ and $A_g(u_g)$, i.e.,

$$\operatorname{int} A_q(u_q) \bigcap \operatorname{int} A_g(u_g) = \varnothing \text{ for } q < g \in I_n, \tag{3}$$

containment of each convex polygon $A_q(u_q)$ into $\Omega(\mathbf{p})$, i.e.,

$$A_q(u_q) \subset \Omega(\mathbf{p}) \Leftrightarrow \operatorname{int} A_q(u_q) \bigcap \Omega^*(\mathbf{p}) = \varnothing \text{ for } q \in I_n, \tag{4}$$

where $\Omega^*(\mathbf{p}) = R^2 \setminus \operatorname{int} \Omega(\mathbf{p})$.

To formalize the placement constraints (3)-(4), the phi-function technique is used to describe analytically relations between a pair of objects.

3 MATHEMATICAL MODELLING

For the reader convenience a few basic defenitions of the phi-function technique are provided.

Let A be a two-dimensional object. The position of the object A is defined by a vector of placement parameters $u_A = (v_A, \theta_A)$, where $v_A = (x_A, y_A, z_A)$ is a translation vector and θ_A is a vector of rotation parametrs. The object A, rotated by θ_A and translated by v_A is denoted by $A(u_A)$.

For two objects $A(u_A)$ and $B(u_B)$ a phi-function allows distinguishing the following three cases: b) $A(u_A)$ and $B(u_B)$ do not overlap, i.e., $A(u_A)$ and $B(u_B)$ do not have any common points; c) $A(u_A)$ and $B(u_B)$ are in contact, i.e., $A(u_A)$ and $B(u_B)$ have only common frontier points; a) $A(u_A)$ and $B(u_B)$ are overlapping so that $A(u_A)$ and $B(u_B)$ have common interior points.

By the definition (Chernov N., Stoyan Yu, Romanova T., 2010) a continuous and everywhere defined function, denoted by $\Phi^{AB}(u_A, u_B)$, is called a phi-function of the objects $A(u_A)$ and $B(u_B)$ if the following conditions are fulfilled:

$$\Phi^{AB}(u_A, u_B) > 0, \text{ for } A(u_A) \cap B(u_B) = \varnothing \ ,$$

$$\Phi^{AB}(u_A,u_B)=0, \text{ for } \operatorname{int} A(u_A)\cap \operatorname{int} B(u_B)=\varnothing \text{ and } frA(u_A)\cap frB(u_B)\neq\varnothing;$$

$$\Phi^{AB}(u_A,u_B)<0, \text{ for } \operatorname{int} A(u_A)\cap \operatorname{int} B(u_B)\neq\varnothing.$$

Here frA denotes the boundary (frontier) of the object A, while $\operatorname{int} A$ stands for its interior.

Thus,

$$\Phi^{AB}(u_A,u_B)\geq 0 \Leftrightarrow \operatorname{int} A(u_A)\cap \operatorname{int} B(u_B)=\varnothing.$$

Define a function $\Phi'^{AB}(u_A,u_B,u')$ introducing auxiliary variables u' defined in a domain $U\subset R^n$ depending on the shapes of the objects A and B. This function is defined for all values of u_A, u_B and has to be continuous in all its variables.

By the definition (Stoyan, Yu., Pankratov, A., Romanova, T., 2016) the function $\Phi'^{AB}(u_A,u_B,u')$ is called a quasi-phi-function for two objects $A(u_A)$ and $B(u_B)$ if $\max\limits_{u'\in U}\Phi'^{AB}(u_A,u_B,u')$ is a phi-function for the objects.

The definition of the quasi-phi-function provides an additional "degree of freedom" since the auxiliary variables u' can be chosen as necessary.

The general property of the quasi-phi-function for two objects $A(u_A)$ and $B(u_B)$ is as follows:

if $\Phi'^{AB}(u_A,u_B,u')\geq 0$ for some u', then $\operatorname{int} A(u_A)\cap \operatorname{int} B(u_B)=\varnothing$.

3.1 Non-overlapping condition

The condition (3) can be stated in twofold: a) using phi-functions; b) using quasi phi-functions.

Let two convex polygons A_q and A_g be defined by their vertices $\tilde{v}_{qi}=(\tilde{x}_{qi},\tilde{y}_{qi})$, $i=1,\ldots,m_q$, and $\tilde{v}_{gj}=(\tilde{x}_{gj},\tilde{y}_{gj})$, $j=1,\ldots,m_g$ respectively. Placement parameters of A_q and A_g are denoted by $u_q=(x_q,y_q,\theta_q)$ and $u_g=(x_g,y_g,\theta_g)$.

Phi-function for two convex polygons

The phi-function for two convex polygons $A_q(u_q)$ and $A_g(u_g)$ can be defined as follows:

$$\Phi_{qg}(u_q,u_g)=\max\{\max_{1\le i\le m_q}\min_{1\le j\le m_g}\varphi^{ij}_{qg}(u_q,u_g),\ \max_{1\le j\le m_g}\min_{1\le i\le m_q}\psi^{ji}_{qg}(u_q,u_g)\},\quad (5)$$

where

$$\varphi^{ij}_{qg}(u_q,u_g)=\alpha'_{qi}x''_{gj}+\beta'_{qi}y''_{gj}+\gamma'_{qi},\ \psi^{ji}_{qg}(u_q,u_g)=\alpha''_{gj}x'_{qi}+\beta''_{gj}y'_{qi}+\gamma''_{gj},$$

$$x'_{qi}=\tilde{x}'_{qi}\cdot\cos\theta_q+\tilde{y}'_{qi}\cdot\sin\theta_q+x_q,\ y'_{qi}=-\tilde{x}'_{qi}\cdot\sin\theta_q+\tilde{y}'_{qi}\cdot\cos\theta_q+y_q,$$

$$x''_{gj}=\tilde{x}''_{gj}\cdot\cos\theta_g+\tilde{y}''_{gj}\cdot\sin\theta_g+x_g,\ y''_{gj}=-\tilde{x}''_{gj}\cdot\sin\theta_g+\tilde{y}''_{gj}\cdot\cos\theta_g+y_g,$$

$$\alpha'_{qi}=\tilde{\alpha}'_{qi}\cdot\cos\theta_q+\tilde{\beta}'_{qi}\cdot\sin\theta_q,\ \ \beta'_{qi}=-\tilde{\alpha}'_{qi}\cdot\sin\theta_q+\tilde{\beta}'_{qi}\cdot\cos\theta_q,$$
$$\gamma'_{qi}=\tilde{\gamma}'_{qi}-\tilde{\alpha}'_{qi}\cdot x_q-\tilde{\beta}'_{qi}\cdot y_q,$$

$$\alpha''_{gj}=\tilde{\alpha}''_{gj}\cdot\cos\theta_g+\tilde{\beta}''_{gj}\cdot\sin\theta_g,\ \ \beta'_{gj}=-\tilde{\alpha}''_{gj}\cdot\sin\theta_g+\tilde{\beta}''_{gj}\cdot\cos\theta_g,$$
$$\gamma''_{gj}=\tilde{\gamma}''_{gj}-\tilde{\alpha}''_{gj}\cdot x_g-\tilde{\beta}''_{gj}\cdot y_g.$$

Therefore $\Phi_{qg}(u_q,u_g)\ge 0$ provides the non-overlapping condition (3), i.e., $\operatorname{int} A_q(u_q)\cap\operatorname{int} A_g(u_g)=\varnothing$.

Quasi phi-function for two convex polygons

Let $P(u_{qg})=\{(x,y):\mu_{qg}=\cos\phi_{qg}\cdot x+\sin\phi_{qg}\cdot y+\gamma_{qg}\le 0\}$ be a half plane, where $u_{qg}=(\phi_{qg},\gamma_{qg})\in R^2$ is a vector of variable parameters of $P(u_{qg})$. Here $\mu_{qg}=0$ is the normal equation of the line with variable coefficients $\cos\phi_{qg}$, $\sin\phi_{qg}$ and free term γ_{qg}.

The quasi phi-function for $A_q(u_q)$ and $A_g(u_g)$ can be derived in the form

$$\Phi'_{qg}(u_q, u_g, u_{qg}) = \min\{\Phi^P_q(u_q, u_{qg}), \Phi^{P^*}_g(u_g, u_{qg})\}, \tag{6}$$

where

$$\Phi^P_q(u_q, u_{qg}) = \min_{1 \le i \le m_q} (\cos\phi_{qg} \cdot x'_{qi} + \sin\phi_{qg} \cdot y'_{qi} + \gamma_{qg})$$

is the normalized phi-function for $A_q(u_q)$ and $P(u_{qg})$,

$$\Phi^{P^*}_g(u_g, u_{qg}) = \min_{1 \le j \le m_g} (-\cos\phi_{qg} \cdot x''_{gj} - \sin\phi_{qg} \cdot y''_{gj} - \gamma_{qg})$$

is the normalized phi-function for $A_g(u_g)$ and $P^*(u_{qg}) = R^2 \setminus \mathrm{int}\, P(u_{qg})$.

Therefore, the inequality $\Phi'_{qg}(u_q, u_g, u_{qg}) \ge 0$ for some $u_{qg} = (\phi^*_{qg}, \gamma^*_{qg})$ provides the non-overlapping condition (3), i.e., $\mathrm{int}\, A_q(u_q) \cap \mathrm{int}\, A_g(u_g) = \varnothing$. In other words, if objects $\mathrm{int}\, A_q(u_q)$ and $\mathrm{int}\, A_g(u_g)$ do not overlap then there is always exists the separation line with the normal equation $\mu_{qg}(\phi^*_{qg}, \gamma^*_{qg}) = 0$.

3.2 Containment conditions

By the definition, $A_q(u_q) \subseteq \Omega(\mathbf{p})$ if $\mathrm{int}\, A_q(u_q) \cap \Omega^* = \varnothing$, $\Omega^* = R^2 \setminus \mathrm{int}\, \Omega$. Below a phi-function $\Phi^{\Omega^* A}(u_A, \mathbf{p})$ for a convex polygon $A_q(u_q)$ and the object $\Omega^*(\mathbf{p})$ is presented.

Let $(\tilde{x}'_{qi}, \tilde{y}'_{qi})$, $i = 1, \ldots, m_q$, be the vertices of A_q and let $\varpi_k = 0$ be an equation of the side $e_k = [\omega_k, \omega_{k+1}]$ of the container Ω, where $\varpi_k = \alpha^\omega_k x + \beta^\omega_k y + \gamma^\omega_k$, such that $\varpi_k(0,0) > 0$.

The container side e_k has its vertices $\omega_k = (x^\omega_k, y^\omega_k)$ and $\omega_{k+1} = (x^\omega_k + t_k \cos\Theta_k, y^\omega_k + t_k \sin\Theta_k)$, while the coefficients of the side equation are $\alpha^\omega_k = -\sin\Theta_k$, $\beta^\omega_k = \cos\Theta_k$ and $\gamma^\omega_k = x^\omega_k \sin\Theta_k - y^\omega_k \cos\Theta_k$ respectively.

The phi-function for a convex polygon $A_q(u_q)$ and the object $\Omega^*(\mathbf{p})$ can be stated in the form

$$\Phi^*(u_q, \mathbf{p}) = \min_{1 \le k \le m} \min_{1 \le i \le m_q} \varpi'_{kqi}(u_q, \mathbf{p}), \tag{7}$$

where

$$\varpi'_{kqi} = \alpha_k^{\omega} x'_{qi} + \beta_k^{\omega} y'_{qi} + \gamma_k^{\omega},$$

and

$$x'_{qi} = \tilde{x}'_{qi} \cdot \cos\theta_q + \tilde{y}'_{qi} \cdot \sin\theta_q + x_q, \qquad y'_{qi} = -\tilde{x}'_{qi} \cdot \sin\theta_q + \tilde{y}'_{qi} \cdot \cos\theta_q + y_q,$$

$i = 1, \ldots, m_q$, are vertices of $A_q(u_q)$.

4 MATHEMATICAL MODEL

Let $\mathbf{p} = (x_1^{\omega}, y_1^{\omega}, \Theta_1, t_1, \ldots, x_m^{\omega}, y_m^{\omega}, \Theta_m, t_m)$ be a vector of variable metrical characteristics of the container Ω, while $(u_1, \ldots, u_n)$ be a vector of variable placement parameters of polygons A_q for $q \in I_n$.

4.1 Mathematical model using phi functions

The problem of finding the minimum perimeter container for N convex polygons can be stated using phi-functions as follows:

$$\min F(u) \quad \text{s.t. } u \in W \tag{8}$$

$$W = \{u \in R^{\sigma} : \Phi_{qg}(u_q, u_g) \ge 0, q < g \in I_n, \Phi^*(u_q, \mathbf{p}) \ge 0, q \in I_n, f(\mathbf{p}) \ge 0\}, \tag{9}$$

where $F(u) = \sum_{i=1}^{m} t_i$ (the perimeter of $\Omega(\mathbf{p})$,

$u = (\mathbf{p}, u_1, \ldots, u_n) =$ $(x_1^{\omega}, y_1^{\omega}, \Theta_1, t_1, \ldots, x_m^{\omega}, y_m^{\omega}, \Theta_m, t_m, u_1, \ldots, u_n) \in R^{\sigma}$ is a vector of variables, R^{σ} is Euclidean space of σ dimension, $\sigma = 4m + 3n$ (for rotatable polygons), $\sigma = 4m + 2n$ (for non-rotatable polygons),

W denotes the corresponding set of feasible solutions (the solution space),

$\Phi_{qg}(u_q, u_g)$ is the phi-function for two convex polygons $A_q(u_q)$ and $A_g(u_g)$,

$\Phi_{qg}(u_q, u_g) \geq 0$ implies $\operatorname{int} A_q(u_q) \bigcap \operatorname{int} A_g(u_g) = \varnothing$;

$\Phi_q^*(u_q, \mathbf{p})$ is the phi-function for the convex polygon $A_q(u_q)$ and the object $\Omega^*(\mathbf{p})$, $\Phi_q^*(u_q, \mathbf{p}) \geq 0$ implies $A_q(u_q) \subseteq \Omega(\mathbf{p})$;

$f(\mathbf{p}) \geq 0$ describes condition (2).

The phi-functions in (9) are composed of $\max$- and $\min$- combinations of linear and\or non-linear functions including $\sin$- and $\cos$-terms (see (5) and (7)).

As a result, the set W of feasible solutions is non-convex, leading to many local extrema. Hence, the *problem* (8)-(9) is a *nonsmooth nonconvex optimization problem*.

A feasible region W of the problem (9)-(10) is presented as a union of subregions $W_s, s = 1, 2, \ldots, \eta$. (For example, $\eta = m_A + m_B$ for two convex polygons, where m_A is the number of vertices of A and m_B is the number of vertices of B, in particular, $\eta = 6$ for two triangles).

This property of the set W comes from the following considerations.

A single inequality

$$\min\{\chi_1(u), \chi_2(u), \ldots, \chi_\eta(u)\} \geq 0$$

is equivalent to the system of η inequalities

$$\begin{cases} \chi_1(u) \geq 0 \\ \chi_2(u) \geq 0 \\ \ldots \\ \chi_\eta(u) \geq 0 \end{cases}.$$

From the inequality

$$\max\{\chi_1(u), \chi_2(u), \ldots, \chi_\eta(u)\} \geq 0$$

it follows that at least one of the functions $\chi_1(u), \chi_2(u), \ldots, \chi_\eta(u)$ is non-negative.

Since $W = \bigcup_{s=1}^{\eta} W_s$ the nonsmooth optimization problem (8)-(9) can be reduced to the following problem for $u = (\mathbf{p}, u_1, \ldots, u_n)$:

$$F(u^*) = \min\{F(u^{s*}), s = 1, \dots, \eta\}, \tag{10}$$

where

$$F(u^{s*}) = \min F(u) \text{ s.t. } u \in W_s. \tag{11}$$

Clearly, the global optimum solution $F(u^*)$ can be obtained and proved by inspecting and exactly solving all subproblems defined in (11).

Subproblems (11) are in general nonlinear mathematical programming problems and they can be solved by standard techniques of local optimization.

4.2 Mathematical model with quasi phi functions

Using the quasi phi-functions the problem of finding the minimum perimeter polygonal container for n convex polygons can be formulated as the following NLP problem.

The decision variables are as follows:

– $\mathbf{p} = (x_1^\omega, y_1^\omega, \Theta_1, t_1, \dots, x_m^\omega, y_m^\omega, \Theta_m, t_m)$ is a variable vector of the container Ω, $k = 1, \dots, m$,

– $u_q = (x_q, y_q, \theta_q)$, $q \in I_n$ are vectors of placement parameters of convex polygons $A_q(u_q)$, $q \in I_n$;

– u_{qg}, $q < g \in I_n$ are vectors of the auxiliary variables in the quasi phi-functions $\Phi'_{qg}(u_q, u_g, u_{qg}), q < g \in I_n$.

Now the packing problem can be formulated in the following from:

$$\min F(u) \text{ s.t. } (u, \tau, \mathbf{p}) \in W', \tag{12}$$

$$W' = \{(u, \tau, \mathbf{p}) : \Phi'_{qg}(u_q, u_g, u_{qg}) \ge 0, q < g \in I_n, \Phi^*_q(u_q, \mathbf{p}) \ge 0, q \in I_n, f(\mathbf{p}) \ge 0\}, \tag{13}$$

where $F(u) = \sum_{i=1}^{m} t_i$ (the perimeter of Ω),

$u = (u_1, \dots, u_n)$, $\tau = (u_{11}, u_{12}, \dots, u_{qg}, \dots, u_{(n-1),n})$,

$\Phi'_{qg}(u_q, u_g, u_{qg}) \geq 0$ assures the non-overlapping constraint (3) for $A_q(u_q)$ and $A_g(u_g)$, $q < g \in I_n$;

$\Phi^*_q(u_q, \mathbf{p}) \geq 0$ represents the containment constraint (4) for $A_q(u_q)$ and Ω^*, $q \in I_n$;

$f(\mathbf{p}) \geq 0$ describes constraints (2) for variable sizes of the container $\Omega(\mathbf{p})$.

The feasible region W given by (13) is defined by a system of non-smooth inequalities that can be reduced to a system of inequalities with smooth functions. This can be done due to the specific type of quasi phi-functions (6) and phi-functions for containment constraints involved in (13).

The model (12)–(13) is a non-convex and continuous nonlinear programming problem. This is an exact formulation in the sense that it gives all optimal solutions to the packing problem.

There are number of the problem variables is $\sigma = 4m + 3n + (n-1)n$. The model (12)–(13) involves $O(n^2)$ nonlinear inequalities.

5 SOLUTION STRATEGY FOR TWO TRIANGLES

5.1 Solution strategy using phi functions

Let A and B be two triangles given by their vertices. And let $\tilde{v}'_i = (\tilde{x}'_i, \tilde{y}'_i)$, $i = 1, 2, 3$, denote the vertices of A, and $\tilde{v}''_j = (\tilde{x}''_j, \tilde{y}''_j)$, $j = 1, 2, 3$, denote those of B. The following presentation of the triangles is use: $A = \{(\tilde{x}, \tilde{y}) : \tilde{\varphi}_i(\tilde{x}, \tilde{y}) \leq 0,\ i = 1, 2, 3\}$, where $\tilde{\varphi}_i(\tilde{x}, \tilde{y}) = \tilde{\alpha}'_i x + \tilde{\beta}'_i y + \tilde{\gamma}'_i$, $\tilde{\alpha}'^2_i + \tilde{\beta}'^2_i = 1$, and $\tilde{\varphi}_i = 0$ is the normal equation of the i-th side of A for $i = 1, 2, 3$;

$B = \{(\tilde{x}, \tilde{y}) : \tilde{\psi}_j(\tilde{x}, \tilde{y}) \leq 0,\ j = 1, 2, 3\}$, $\tilde{\psi}_j(\tilde{x}, \tilde{y}) = \tilde{\alpha}''_j x + \tilde{\beta}''_j y + \tilde{\gamma}''_j$, $\tilde{\alpha}''^2_j + \tilde{\beta}''^2_j = 1$, and $\tilde{\psi}_j = 0$ is the normal equation of the j-th side of B for $j = 1, 2, 3$.

To break symmetry and degeneration assume that triangle A is fixed and triangle B can freely be translated and rotated.

The phi-function (5) for the triangles A and $B(u_B)$ takes the form:

$$\Phi^{AB}(u_B) = \max\{\max_{1\le i\le 3}\min_{1\le j\le 3}\varphi_{ij}(u_B), \max_{1\le j\le 3}\min_{1\le i\le 3}\psi_{ji}(u_B)\}, \qquad (14)$$

where $\varphi_{ij}(u_B) = \tilde{\alpha}'_i x''_j + \tilde{\beta}'_i y''_j + \tilde{\gamma}'_i$, $\psi_{ji}(u_B) = \alpha''_j \tilde{x}'_i + \beta''_j \tilde{y}'_i + \gamma''_j$,

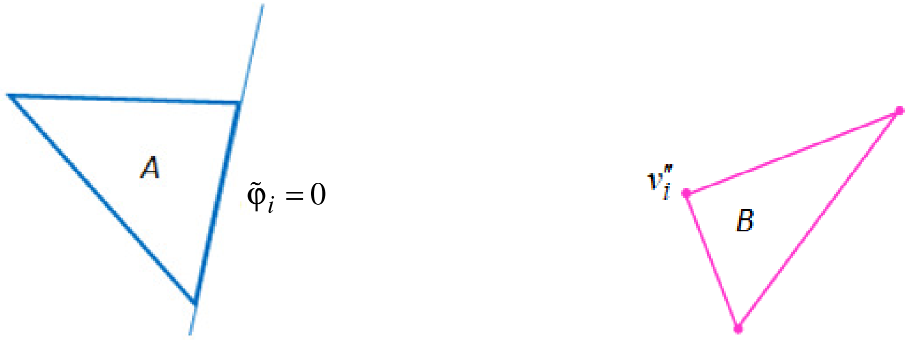
$\varphi_{ij}(u_B) = \tilde{\varphi}_i(x''_j, y''_j)$, $\psi_{ji}(u_B) = \tilde{\psi}_j(\alpha''_j, \beta''_j, \gamma''_{ij})$.

To simplify notations, φ_{ij} will be used for $\varphi_{ij}(u_B)$ and ψ_{ji} for $\psi_{ji}(u_B)$.

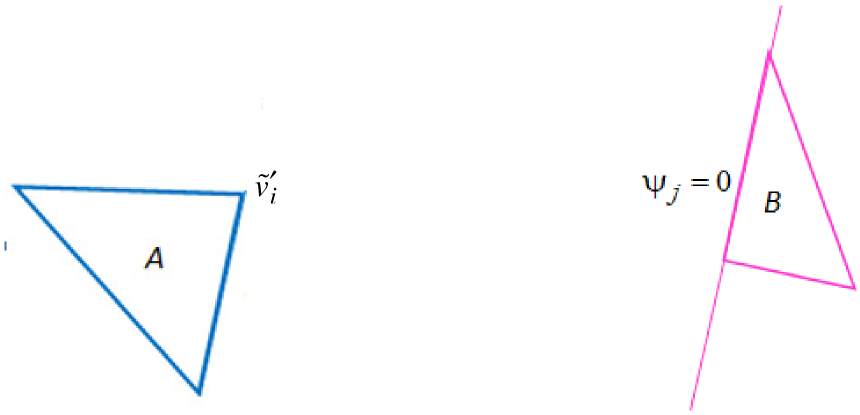
It follows from (14) that for a fixed u_A

$$\Phi^{AB}(u_B) \ge 0 \text{ if}$$

$\max_{1\le i\le 3}\min_{1\le j\le 3}\varphi_{ij} \ge 0$ (fixed sides of triangle A and variable vertices of triangle $B(u_B)$, case 1, Fig. 3a) or $\max_{1\le j\le 3}\min_{1\le i\le 3}\psi_{ji} \ge 0$ (variable sides of triangle $B(u_B)$ and fixed vertices of triangle A, case 2, Fig. 3b).



a)



b)

Fig. 3 Two cases are meeting in Φ^{AB}: a) case 1; b) case 2

Moreover,

$$\max_{1\le i\le 3}\min_{1\le j\le 3}\varphi_{ij}\ge 0 \quad \text{if} \quad \min_{1\le j\le 3}\varphi_{1j}\ge 0 \text{ or } \min_{1\le j\le 3}\varphi_{2j}\ge 0 \text{ or } \min_{1\le j\le 3}\varphi_{3j}\ge 0,$$

$$\max_{1\le j\le 3}\min_{1\le i\le 3}\psi_{ji}\ge 0 \text{ if } \min_{1\le i\le 3}\psi_{1i}\ge 0 \text{ or } \min_{1\le i\le 3}\psi_{2i}\ge 0 \text{ or } \min_{1\le i\le 3}\psi_{3i}\ge 0,$$

where

$\min_{1\le j\le 3}\varphi_{1j}\ge 0$ can be replaced by the system

$$\begin{cases}\varphi_{11}\ge 0\\ \varphi_{12}\ge 0\\ \varphi_{13}\ge 0\end{cases} \tag{15}$$

Simililary for the other nonosmoth inequalities $\min_{1\le j\le 3}\varphi_{2j}\ge 0$, $\min_{1\le j\le 3}\varphi_{3j}\ge 0$, $\min_{1\le i\le 3}\psi_{1i}\ge 0$, $\min_{1\le i\le 3}\psi_{2i}\ge 0$, $\min_{1\le i\le 3}\psi_{3i}\ge 0$ we define the appropriate systems

$$\begin{cases}\varphi_{21}\ge 0\\ \varphi_{22}\ge 0,\\ \varphi_{23}\ge 0\end{cases}\begin{cases}\varphi_{31}\ge 0\\ \varphi_{32}\ge 0,\\ \varphi_{33}\ge 0\end{cases}\begin{cases}\psi_{11}\ge 0\\ \psi_{12}\ge 0,\\ \psi_{13}\ge 0\end{cases}\begin{cases}\psi_{21}\ge 0\\ \psi_{22}\ge 0,\\ \psi_{23}\ge 0\end{cases}\begin{cases}\psi_{31}\ge 0\\ \psi_{32}\ge 0.\\ \psi_{33}\ge 0\end{cases}$$

Finally, we have six inequality systems (15)-(20).

Note that for the containment conditions the phi-function (7) is used, therefore

$$\Phi^{A\Omega^*}(\mathbf{p}) = \min_{1\le k\le m}\min_{1\le i\le 3}\varpi'_{ki}(\mathbf{p}),$$

$$\Phi^{B\Omega^*}(u_B,\mathbf{p}) = \min_{1\le k\le m}\min_{1\le j\le 3}\varpi''_{ki}(u_B,\mathbf{p}).$$

Thus for $m = m_A + m_B = 6$

$$\Phi^{A\Omega^*}\ge 0 \Leftrightarrow \begin{cases}\omega'_{11}\ge 0\\ \omega'_{12}\ge 0\\ \omega'_{13}\ge 0\\ \ldots\\ \omega'_{61}\ge 0\\ \omega'_{62}\ge 0\\ \omega'_{63}\ge 0\end{cases} \text{ and } \Phi^{B\Omega^*}\ge 0 \Leftrightarrow \begin{cases}\omega''_{11}\ge 0\\ \omega''_{12}\ge 0\\ \omega''_{13}\ge 0\\ \ldots\\ \omega''_{61}\ge 0\\ \omega''_{62}\ge 0\\ \omega''_{63}\ge 0\end{cases}. \tag{16}$$

Now the problem (8)-(9) for two triangles A and $B(u_B)$ with the feasible region

$$W = W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5 \cup W_6$$

can be reduced to the problem (10)-(11) of the form

$$F(u^*) = \min\{F(u^{s*}), s = 1, \ldots, 6\},$$

$$F(u^{s*}) = \min F(u) \text{ s.t. } u \in W_s,$$

where the objective function to be minimized is simply

$$F(u) = \sum_{i=1}^{6} t_i,$$

$u = (\mathbf{p}, u_B) = (x_1, y_1, \theta_1, t_1, \ldots, x_6, y_6, \theta_6, t_6, u_B) \in R^\sigma$ is a vector of variables, $\sigma = 4m + 3 = 24 + 3 = 27$ is the number of variables, $\mathbf{p} = (x_1, y_1, \theta_1, t_1, \ldots, x_6, y_6, \theta_6, t_6)$ is a vector of variable metrical characteristics of the container Ω, $u_B = (x_B, y_B, \theta_B)$ is a vector of variable placement parameters of triangle B.

Each W_s is defined by the system (16) describing the containment constrants combained with one of the inequality systems of the form (15) describing the non-overlapping conditions, $s = 1, 2, \ldots, 6$, and the inequality system $f \geq 0$ describing condition (2).

5.2 Solution strategy using quasi phi functions

The quasi phi-function (7) for the triangles A and $B(u_B)$ takes the form:

$$\Phi'_{AB}(u_B, u_{AB}) = \min\{\Phi_A^P(u_{AB}), \Phi_B^{P^*}(u_B, u_{AB})\},$$

where

$$\Phi_A^P(u_{AB}) = \min_{1 \leq i \leq 3}(\cos\phi_{AB} \cdot \tilde{x}'_{Ai} + \sin\phi_{AB} \cdot \tilde{y}'_{Ai} + \gamma_{AB})$$

is the normalized phi-function for A and $B(u_{AB})$,

$$\Phi_B^{P^*}(u_B, u_{AB}) = \min_{1\le j\le 3}(-\cos\phi_{AB}\cdot x''_{Bj} - \sin\phi_{AB}\cdot y''_{Bj} - \gamma_{AB})$$

Therefore for describing the non-overlapping condition, the nonsmooth inequality

$$\Phi'_{AB}(u_B, u_{AB}) \ge 0$$

is reduced equivalently to the following system of *six* inequalities with smooth functions

$$\begin{cases} \cos\phi_{AB}\cdot x'_{11} + \sin\phi_{AB}\cdot y'_{11} + \gamma_{AB} \ge 0 \\ \cos\phi_{AB}\cdot x'_{12} + \sin\phi_{AB}\cdot y'_{12} + \gamma_{AB} \ge 0 \\ \cos\phi_{AB}\cdot x'_{13} + \sin\phi_{AB}\cdot y'_{13} + \gamma_{AB} \ge 0 \\ -\cos\phi_{AB}\cdot x''_{21} - \sin\phi_{AB}\cdot y''_{21} - \gamma_{AB} \ge 0 \\ -\cos\phi_{AB}\cdot x''_{22} - \sin\phi_{AB}\cdot y''_{22} - \gamma_{AB} \ge 0 \\ -\cos\phi_{AB}\cdot x''_{23} - \sin\phi_{AB}\cdot y''_{23} - \gamma_{AB} \ge 0 \end{cases}$$

Note that for describing the containment conditions the system (16) with 36 inequalities is used:

$$\begin{cases} \omega'_{11} \ge 0 \\ \omega'_{12} \ge 0 \\ \omega'_{13} \ge 0 \\ \omega''_{11} \ge 0 \\ \omega''_{12} \ge 0 \\ \omega''_{13} \ge 0 \\ \dots \\ \omega'_{61} \ge 0 \\ \omega'_{62} \ge 0 \\ \omega'_{63} \ge 0 \\ \omega''_{61} \ge 0 \\ \omega''_{62} \ge 0 \\ \omega''_{63} \ge 0 \end{cases}$$

Now the packing problem of two triangles for $m = m_A + m_B = 6$ can be formulated as the following nonlinear optimization problem:

$$\max \sum_{i=1}^{6} t_i \text{ s.t. } \left(u_B, u_{AB}, \mathbf{p}\right) \in W',$$

$$W' = \{(u_B, u_{AB}, \mathbf{p}) : \Phi'_{AB}(u_B, u_{AB}) \geq 0, \Phi^*_A(\mathbf{p}) \geq 0, \Phi^*_B(u_B, \mathbf{p}) \geq 0, f(\mathbf{p}) \geq 0\}$$

$\Phi'_{AB}(u_B, u_{AB}) \geq 0$ represents the non-overlapping constraint (3) for triangles A and $B(u_B)$;

$\Phi^*_A(\mathbf{p}) \geq 0$ represents the containment constraint (4) for A and Ω^*;

$\Phi^*_B(u_B, \mathbf{p}) \geq 0$ describes the containment constraint (4) for $B(u_B)$ and Ω^*;

$f(\mathbf{p}) \geq 0$ means auxiliary conditions for variable sizes of the container $\Omega(\mathbf{p})$.

The variables used in the optimization problem are as follows:

$u = (\mathbf{p}, u_A, u_B) = (x_1, y_1, \theta_1, t_1, \ldots, x_6, y_6, \theta_6, t_6, u_B, u_{AB}) \in R^{\sigma}$ is a vector of variables, $\sigma = 4m + 3 + 2 = 24 + 5 = 29$ is the number of variables;

$\mathbf{p} = (x_1, y_1, \theta_1, t_1, \ldots, x_6, y_6, \theta_6, t_6)$ is a vector of variable metrical characteristics of the container Ω;

$u_B = (x_B, y_B, \theta_B)$ is a vector of variable placement parameters of triangle $B(u_B)$;

$u_{AB} = (\phi_{AB}, \gamma_{AB})$ is the vector of the auxiliary variables in the quasi phi-function $\Phi'_{AB}(u_B, u_{AB})$;

the feasible region W' is described by the system of 42 inequalities with smooth functions.

6 COMPUTATIONAL RESULTS

In this section numerical examples are presented. The problem instances 1-6 were solved to optimality by the global NLP solver BARON (Khajavirad 2018, Sahinidis 2019, Tawarmalani 2005). These experiments were run on a 64 bit machine with an Intel(R) Core(TM) i7 CPU 2.8 GHz, 16 GB, RAM, Windows 7. Numerical results for larger instances 7, 8 were obtained by the non-commercial local NLP solver IPOPT (https://github.com/coin-or/Ipopt) developed in (Wächter, A., Biegler, L.T., 2006).

Default options were used for running this software. These experiments were run on an AMD FX(tm)-6100, 3.30 GHz computer, Programming Language C++, Windows 7. The CPU time limit for running BARON in examples 1-6 was set to 48 hours, while for running IPOPT in examples 7 and 8 the time limit was 12 min.

For each problem instance m^* ($m^* \leq m$) denotes the number of vertices (sides) in the minimal perimeter container Ω. Note, that the inequality $m^* \leq m$ can be strict since by optimization some vertices may coinside and the length of the corresponding side can be zero. The value of the minimal perimeter is denoted by F^*.

6.1 Examples for the phi-function model

Example 1. Two triangles given by their vertices are considered: triangle 1 = {(0, 0) (14, 0) (10, -5)}, triangle 2 = {(0, 0) (8, 0) (6, 4)}. For $m = 6$ (the total number of vertices in 2 triangles) the minimum perimeter convex hull with $F^* = 33.707980000000$ and $m^* = 4$ was found by BARON.

Figure 4 shows the configuration for the two triangles that corresponds to the global solution **.**

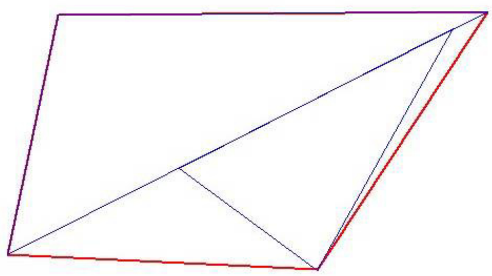


Fig. 4 The minimum perimeter convex hull for the two triangles

Example 2. Two objects, triangle and quadrangle are considered given by their vertices: triangle ={(0, 0) (8, 0) (6, 4)}, quadrangle ={(0, 0) (7.5, 0) (2, 4) (-5, 5)}. For $m = 7$ (the total number of vertices in triangle and quadrangle) the minimum perimeter convex hull with $F^* = 31.8680963110139$ and $m^* = 4$ was found by BARON. Figure 5 shows the configuration for the triangle and the quadrangle that corresponds to the global solution **.**

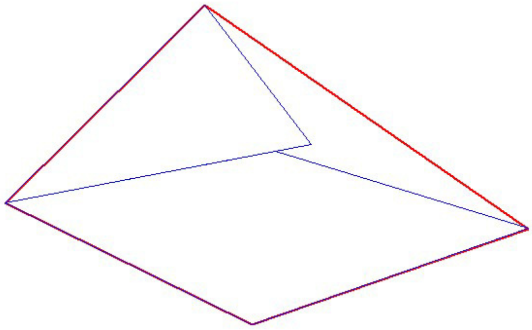


Fig. 5 The minimum perimeter convex hull for triangle and quadrangle

Example 3. Two equal pentagons are considered given by the vertices: pentagon 1 $=\{(1, 0),\ (-1, 0),\ (-9, 1),\ (0, 10),\ (9, 1)\}$, pentagon 2 $=\{(1, 0),\ (-1, 0),\ (-9, 1),\ (0, 10),\ (9, 1)\}$. For $m=10$ (the total number of vertices in 2 pentagons) the minimum perimeter convex hull with $F^{*}=54.911688000000$ and $m^{*}=6$ was found by BARON.

Figure 6 shows the configuration for the two pentagons that corresponds to the global solution **.**

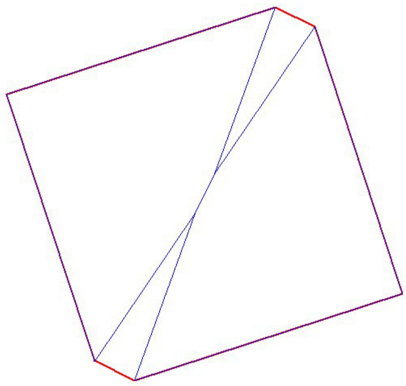


Fig. 6 The minimum perimeter convex hull for two pentagons

Example 4. Three triangles are considered given by their vertices: triangle 1 $=\{(0, 0)\ (4, 3)\ (3, 0)\}$, triangle 2 $=\{(0, 0)\ (4, -3)\ (3, 0)\}$, triangle 3 $= \{(0, 0)\ (1, -3)\ (1, 3)\}$. For $m=9$ (the total number of vertices in 3 triangles) the minimum perimeter convex hull with $F^{*}=15.1790222006124$ and $m^{*}=5$ was found by BARON.

Figure 7 shows the configuration for two pentagons that corresponds to the global solution **.**

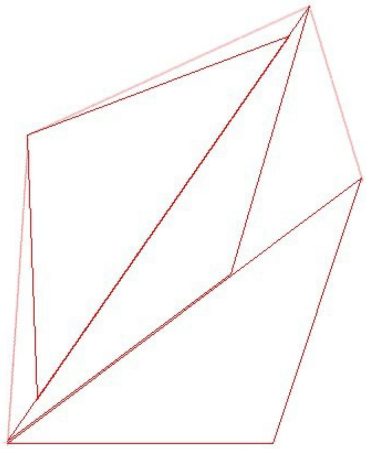


Fig. 7 The minimum perimeter convex hull for three triangles

Example 5. Four quadrangles are considered given by their vertices: quadrangle 1 ={(0, 0) (0, 4) (2, 4) (5, 0)}, quadrangle 2 ={(0, 0) (0, 4) (2, 4) (5, 0)}, quadrangle 3 = {(0, 0) (0, -4) (2,-4) (5, 0)}, quadrangle 4 ={(0, 0) (0, -4) (2,-4) (5, 0)}. For $m = 16$ (the total number of vertices in 4 quadrangles) the minimum perimeter convex hull with $F^* = 28.000000000000$ and $m^* = 6$ was found by BARON. Figure 8 shows the configuration for four quadrangles that corresponds to the global solution**.**

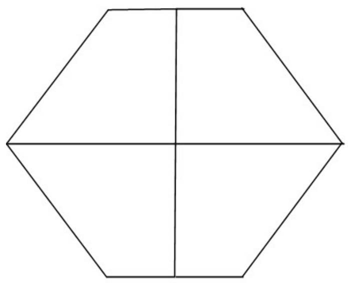


Fig. 8 The minimum perimeter convex hull for four quadrangles

Example 6. Six equal triangles are considered given by the vertices: $\{(0,\ 0)\ (0, 4)\ (2, 4)\}$. For $m = 18$ (the total number of vertices in 6 triangles) the minimum perimeter convex polygonal container with $F^* = 19.416375209619$ and $m^* = 6$ was found by BARON. Figure 9 shows the optimal configuration for six triangles.

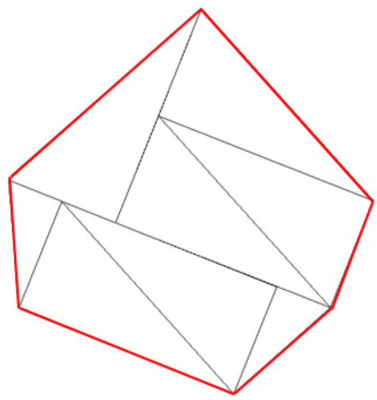


Fig. 9 The optimized perimeter polygonal container for six triangles

6.2 Examples for the quasi phi-function model

Example 7. Ten regular pentagons inscribed in circles of the given radii are considered. The corresponding radii for the pentagons 1, 2 are $r_1 = r_2 = 4$, for pentagons 3, 4: $r_3 = r_4 = 3$, for pentagons 5, 6, 7: $r_5 = r_6 = r_7 = 2$, for pentagons 8, 9, 10: $r_8 = r_9 = r_{10} = 1$. For $m = 12$ the locally minimal perimeter convex polygonal container with $F^* = 49.2339$ and $m^* = 8$ was found by IPOPT. Figure 10 shows the configuration for ten pentagons that corresponds to the local optimal solution**.**

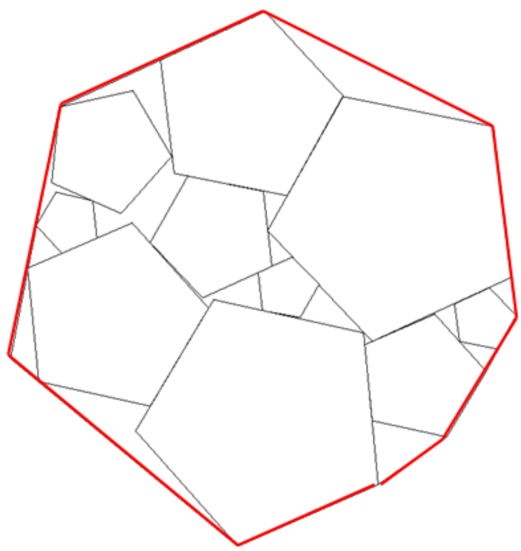


Fig. 10 The optimized perimeter polygonal container for ten pentagons

Example 8. Twenty five convex polygons are considered given by their vertices: 13 equal trapezoids = {(0, 0) (8, 0) (14, 7) (14, 15)} and 12 equal triangles = {(0, 0) (11, 0) (14, 11)}. For $m = 20$ the locally minimal perimeter convex polygonal container with $F^* = 166.6851$ and $m^* = 12$ was found by IPOPT. Figure 11 shows the configuration for ten pentagons that corresponds to the local optimal solution**.**

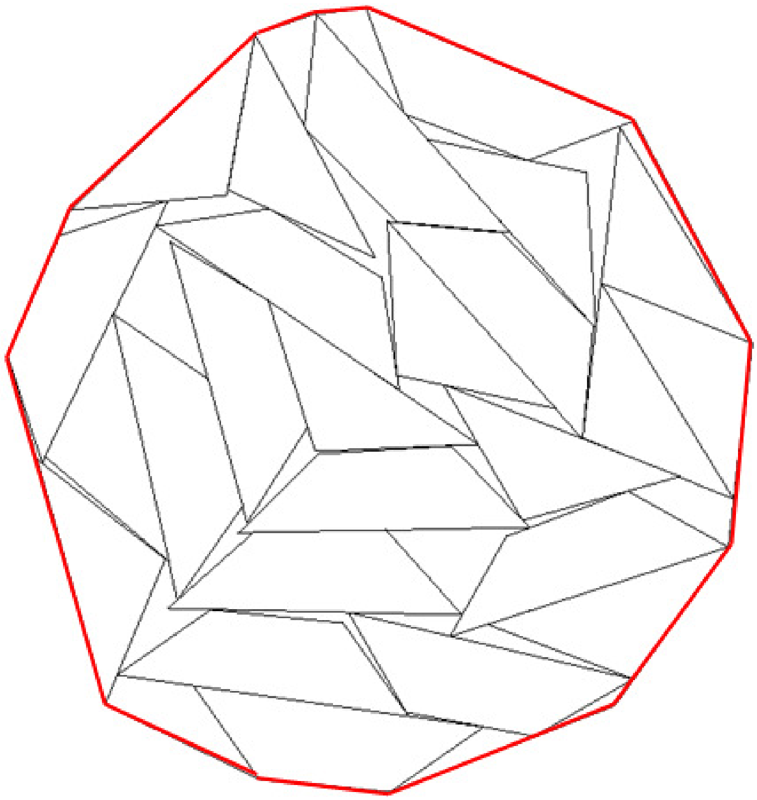


Fig. 11 The optimized perimeter polygonal container for 25 convex polygons

7 CONCLUSIONS AND OUTLOOK

In this paper packing convex polygons in a minimal perimeter convex polygonal container has been considered. The phi-functions approach has been used to state non-overlapping and containment conditions and form the corresponding nonlinear programming problem.

An interesting direction for the future research is the generalization of the proposed approach for packing in a minimum-area convex polygon. In this case convexity conditions for the container should be stated explicitly in the model since they do not follow from the optimality (as in the minimal perimeter problem). Also, balancing and other equilibrium constraints (Wäscher 2007; Grebennik 2018; Gimenez-Palacios 2020) can be taken into account. Additional shape constraints for the container may also be imposed. For example, if the vertices of the container are considered as locations for transmitting antennas, then the maximal distance between the vertices may be bounded to assure the quality of the signal reception. The costs related to the vertices may also be introduced. Results in this direction are on the way.

Packing freely translated and rotated ellipses in an optimized convex polygon of a given shape was considered, e.g., in (Kampas 2019, 2020; Pankratov 2019, 2020). It is interesting to generalize their approach for constructing a minimum perimeter/area convex *m*-gon.

Packing two non-convex objects (bounded by circular lines and/or line segments) in a minimum perimeter/area convex polygon was considered in (Bennel 2015). Layout of non-convex polygons (Jones 2013; Stoyan 2017), or irregular 3D polyhedra (Stoyan 20179), or ellipsoids (Romanova 2020) in a minimal convex polygonal container is the other promising research direction (see also Araújo 2019; Bennell 2008, 2009; Fasano 2014; Leao 2020; Warade 2020 and the references therein).

The approach proposed in this paper for packing polygons in a minimum-perimeter container requires solution of a non-convex mathematical programming problem. In this paper for some small problem instances global solutions have been obtained by the commercial solver BARON, while for medium-sized instances local optima have been calculated by freely available solver IPOPT in a reasonable computation time. To solve larger instances special techniques can be applied, such as decomposition (Litvinchev 2010) or aggregation (Litvinchev 1999). Some results in this direction are on the way.

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Appendix. On an analytical solution for the minimum perimeter convex hull of two triangles.

For the case of two triangles, we may expect geometrical characteristics of the optimal layout. Leaving the rigorous proof for our forthcoming paper, we present here these characteristics in the form of the following conjecture.

Conjecture. The minimal perimeter convex hull for two triangles corresponds to a layout with two tangent largest sides of the triangles (see Fig. 12)

The motivation for this conjecture is as follows. If the triangles have no common points (totally separated), then we can move one triangle till they are tangent thus reducing the perimeter. Suppose that there is only one tangent point (a vertex of the triangle). Then we can rotate the corresponding triangle around its tangent vertex till the triangles have tangent sides and thus reducing the perimeter. Finally, among all layouts with tangent sides, the case of tangent largest sides corresponds to the minimal perimeter convex hull by the triangle inequality. The rigorous proof of this conjecture is on the way.

Suppose that we have two triangles with their largest tangent sides. Then to get a layout with minimum perimeter convex hull we must define the position of the vertex of one triangle on the tangent side of another (see Figure 12). This can be done as follows.

Consider two triangles with vertices A_i, B_i, and C_i, sides a_i, b_i, and c_i, and internal angles α_i, β_i, and γ_i, $i = 1, 2$. The naming is such, that $c_1 = A_1B_1$ is the largest side of the triangle 1 and that c_1 is greater than or equal to the largest side of the triangle 2. Therefore, A_1 and B_1 are vertices of the convex hull polygon P.

Triangle 1 is placed such that its vertex A_1 is in the origin of the $x-y$-coordinate system, B_1 is at $(c, 0)$, and C_1 is in the fourth quadrant with at (x_1, y_1) with $x_1 > 0$ and $y_1 < 0$. C_1 is another vertex of the convex hull polygon P.

If we place triangle 2 such that A_2 and B_2 are on the positive axis with $0 \le x_{2A} < c$ and $0 < x_{2B} < c$, then C_2 at (x_2, y_2) completes the system of vertices of P (Figure 12).

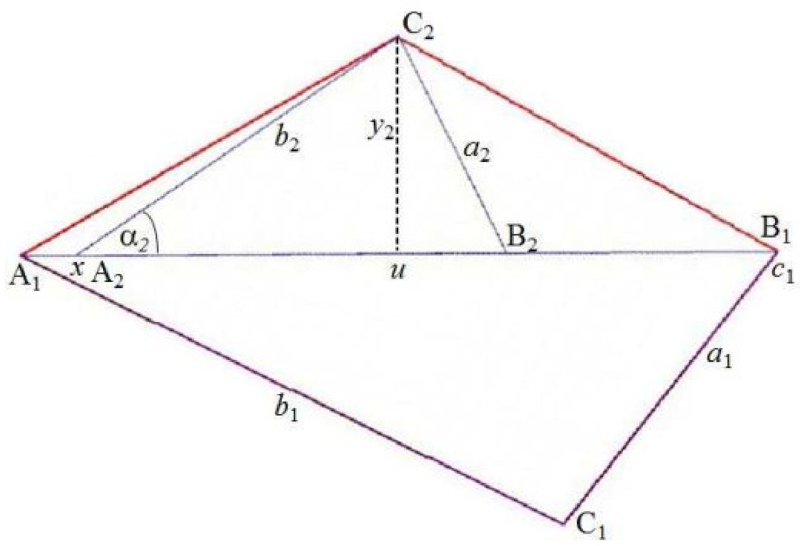

Fig. 12 The minimum perimeter convex hull of two tangent triangles

The y_2-coordinate follows from the triangle itself; it is just its height, i.e.,

$$y_2 = b_2 \sin\alpha_2,\ \alpha_2 = arc\cos\frac{b_2^2 + c_2^2 - a_2^2}{2b_2c_2},$$

where $(v_{x\,2A}, v_{y\,2A}) = (0,0)$, $(v_{x\,2B}, v_{y\,2B})$ and $(v_{x\,2C}, v_{y\,2C})$ are the intrinsic vertex coordinates of triangle 2.

As

$$x_{2B} = x_{2A} + c,$$

and

$$x_{2C} = x_2 = x_{2A} + b_2\cos\alpha_2,$$

the convex hull perimeter ℓ = ℓ(P) depends only on one variable $x = x_{2A}$:

$$l = a_1 + b_1 + \sqrt{(x + b_2\cos\alpha_2)^2 + y_2^2} + \sqrt{(c_1 - (x + b_2\cos\alpha_2))^2 + y_2^2}.$$

The contributions a_1 and b_1 are constant and can be computed a *priori* – for this arrangement of the two triangles. Other arrangements – in which the two triangles have one touching line – follow by permutations and can be evaluated similarly.

For simplification, let us introduce

$$u = x + b_2 \cos \alpha_2 \ , \ v = a_1 + b_1$$

and follow up with

$$l(u) = v + \sqrt{u^2 + y_2^2} + \sqrt{(c_1 - u)^2 + y_2^2}.$$

Now we consider $l'(u)$ and $l''(u)$ to find the local minium u_*:

$$l'(u) = \frac{u}{\sqrt{y_2^2 + u^2}} - \frac{c_1 - u}{\sqrt{y_2^2 + (c_1 - u)^2}}$$

and

$$l''(u) = \frac{1}{\sqrt{u^2 + y_2^2}} + \frac{1}{\sqrt{y_2^2 + (c_1 - u)^2}} - \frac{u^2}{(u^2 + y_2^2)^{\frac{3}{2}}} - \frac{(c_1 - u)^2}{(y_2^2 + (c_1 - u)^2)^{\frac{3}{2}}}.$$

For $u_* = \frac{c_1}{2}$ the first derivative $l'(u_*)$ vanishes. The second derivative takes the value

$$l''(u) = 2\left[\frac{1}{\sqrt{\left(\frac{c_1}{2}\right)^2 + y_2^2}} - \frac{\left(\frac{c_1}{2}\right)^2}{\left(\left(\frac{c_1}{2}\right)^2 + y_2^2\right)^{\frac{3}{2}}}\right] = \frac{2y_2^2}{\left(\left(\frac{c_1}{2}\right)^2 + y_2^2\right)^{\frac{3}{2}}} \geq 0$$

which is always positive, i.e., we have a local minimum at $u_* = \frac{c_1}{2}$, or

$$x = \frac{c_1}{2} - b_2 \cos \alpha_2 = \frac{c_1}{2} - \frac{b_2^2 + c_2^2 - a_2^2}{2c_2}.$$